Semialgebraic groups over real closed fields

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Abstract

This thesis investigates semialgebraically connected semialgebraic groups over a sufficiently saturated real closed field $\mathcal{R} = (R, <, +, 0, \cdot, 1)$, and is therefore a contribution to the study of definable groups in o-minimal structures.

It is proved that for every definably compact definably connected semialgebraic group $G$ over $\mathcal{R}$ there are a Zariski-connected $R$-algebraic group $H$, a definable injective map $\phi$ from a generic definable neighborhood of the identity of $G$ into the group $H(R)$ of $R$-points of $H$ such that $\phi$ acts as a group homomorphism inside its domain.

The above result and our study of locally definable covering homomorphisms for locally definable groups combine to prove that if such a group $G$ is in addition abelian, then its o-minimal universal covering group $\tilde{G}$ is definably isomorphic, as a locally definable group, to a connected open locally definable subgroup of the o-minimal universal covering group $H(R)^0$ of the group $H(R)^0$ for some Zariski-connected $R$-algebraic group $H$. In case we are considering a one-dimensional definably connected group definable in $\mathcal{R}$, then its o-minimal universal covering group is definably isomorphic to either $H(R)^0$ or some open locally definable subgroup of $H(R)^{00}$ for some one-dimensional Zariski-connected $R$-algebraic group $H$. 

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Contents

1 Introduction 8
   1.1 The structure of the thesis 13

2 Preliminaries 16
   2.1 Real closed fields and their definable sets 17
   2.2 $\lor$-definable groups 18
   2.3 Some results of one-dimensional groups definable in o-minimal structures 21
     2.3.1 An orientation on one-dimensional definably compact semialgebraic groups 23
   2.4 On algebraic groups 24
   2.5 Algebraic and semialgebraic groups 26
     2.5.1 Examples of one-dimensional semialgebraic groups 27

3 Group-generic points and generic sets 29
   3.1 Algebraic dimension and generic points 29
   3.2 Generic sets in $\lor$-definable groups 31
3.3 Group-generic points for $V$-definable groups .................................. 32
3.3.1 Basics on group-generic points .................................................. 33
3.4 Generic sets in the product group .................................................. 39

4 Semialgebraically compact semialgebraic groups and algebraic groups over a real closed field 44
4.1 A group configuration proposition for definably compact groups .... 45
4.2 Local homomorphisms ................................................................. 53
4.3 A local homomorphism with generic domain between a semialgebraically compact semialgebraic group over $R$ and the $R$-points of an $R$-algebraic group .................................................. 55

5 Locally definable covering homomorphisms of locally definable groups 61
5.1 Ld-spaces and ld-covering maps .................................................... 62
5.1.1 Ld-spaces and ld-maps .............................................................. 62
5.1.2 Some topological notions in ld-spaces ....................................... 63
5.1.3 Covering maps for ld-spaces .................................................... 66
5.2 The o-minimal universal covering homomorphism of a locally definable group .................................................. 70
5.3 Abelian definably generated groups, convex sets, and covers of definable groups .................................................. 73
5.4 Local homomorphisms and generic sets: some technical propositions . 77
5.5 Extension of a definable local homomorphism from a torsion free abelian locally definable group .................................................. 79
5.6 Universal covers of locally homomorphic abelian locally definable groups 83
5.6.1 The o-minimal universal covering group of an abelian definably connected definably compact semialgebraic group . . . . . . . 84

6 Characterization of the 1-dimensional semialgebraic groups 86

A Proposition 3.1 of Hrushovski and Pillay in [21] 92
Chapter 1

Introduction

Definable groups in o-minimal structures have been intensively studied in the last three decades, and it is a field of current research. A real closed field is an ordered field elementarily equivalent to the real ordered field $\mathbb{R}$; for instance, $\mathbb{R}$, the real algebraic numbers $\mathbb{R}_{\text{alg}}$, the $\aleph_1$-saturated hyperreal numbers $\ast\mathbb{R}$, which has infinite and infinitesimal elements, among other examples. By quantifier elimination in real closed fields (Tarski-Seidenberg), the definable sets in a real closed field $\mathcal{R} = (\mathbb{R}, <, +, 0, \cdot, 1)$ are the semialgebraic sets over $\mathbb{R}$; namely, sets that are finite Boolean combination of sets of solutions of finitely many polynomial equations and inequalities over $\mathbb{R}$.

Since a real closed field is an o-minimal structure (i.e., an ordered structure for which every definable subset of its universe is a finite union of points and intervals, see e.g., [48]), then semialgebraic groups over a real closed field (see Def. 2.1.3) can be seen as a generalization of the semialgebraic groups over the real field $\mathbb{R}$, and also as a particular case of the groups definable in an o-minimal structure.

Let $G$ be a group definable in an o-minimal structure. Pillay proved in [35] that $G$ can be equipped with a unique definable manifold structure making the group into a topological group; we call this topology on $G$ the t-topology. From now on any topological property of definable groups in an o-minimal structure refers to this t-topology.

We say that a definable subset $X \subseteq G$ is definably connected if $X$ has no nonempty proper definable subset (t-)clopen relative to $X$. By [35, Corollary 2.10], there is a unique maximal definably connected definable subset of $G$ containing the identity $e_G$. 
of $G$, which we call the definable identity component of $G$, and we denoted it by $G^0$. Thus, $G$ is definably connected if and only if $G = G^0$, or, equivalently by [35], if $G$ has no proper definable subgroup of finite index. We say that $G$ is definably compact if every definable path $\gamma : (0,1) \to G$ has limits points in $G$ (where the limits are taken with respect to the t-topology on $G$). In case $G$ is a definable group in a real closed field (i.e., a semialgebraic group), we also say that $G$ is semialgebraically connected (semialgebraically compact) to say that $G$ is definably connected (definably compact).

Some contributions to the research on one-dimensional groups definable in an o-minimal structure $\mathcal{M}$ have been offered, for example, by Razenj [39] and Strzebonski [45]. In [39] Razenj classified the one-dimensional definably connected $\mathcal{M}$-definable groups, but up to abstract group isomorphisms, and Strzebonski in [45] dealt with one-dimensional definably connected groups $G \subseteq M^n$ whose group operation is continuous with respect to the product topology of $M^n$ given by the order topology on $(M, <)$, up to $\mathcal{M}$-definable group isomorphisms.

In case the structure $\mathcal{M}$ is the ordered real field $\mathbb{R}$, the t-topology can be endowed an $\mathcal{M}$-definable group $G$ with a structure of real Lie group ([35]), thus $G$ carries a semialgebraic and analytic structure: $G$ is a Nash group (a semialgebraic group over $\mathbb{R}$ that is also real analytic). Using the machinery of Nash groups over $\mathbb{R}$, Madden and Stanton provided in [26] a full characterization of the one-dimensional semialgebraically connected semialgebraic groups over $\mathbb{R}$ up to semialgebraic group isomorphisms.

From now until the end of this introduction, $\mathcal{R} = (\mathbb{R}, <, +, 0, \cdot, 1)$ denotes a real closed field.

We can ask for a description of the one-dimensional definably connected semialgebraic groups over $\mathbb{R}$. We might be tempted to transfer from $\mathbb{R}$ to $\mathcal{R}$ the work of Madden and Stanton of groups definable in $\mathbb{R}$ to obtain an understanding of groups definable in $\mathcal{R}$; nevertheless, there are real closed fields with infinitesimal and infinite elements that can be used as parameters to define a group. For instance, consider the $\mathcal{R}$-definable group $([0,\alpha) \subseteq \mathbb{R}, +_{mod\alpha})$ with addition modulo $\alpha$ defined as $x +_{mod\alpha} y$ is equal to $x + y$ if $x + y < \alpha$ or $x + y - \alpha$ otherwise, where $\mathcal{R}$ is a $\aleph_1$-saturated real closed field and $0 < \alpha < \frac{1}{n}$ for every $n \in \mathbb{N}$.

From now on, by a sufficiently saturated structure we mean a $\kappa$-saturated structure for some sufficiently large cardinal $\kappa$. 

9
This thesis offers a description of the semialgebraically connected semialgebraic groups over a sufficiently saturated real closed field \( R \) through the study of their o-minimal universal covering groups (see Def. 5.2.3) and of their relation with the \( R \)-points of some Zariski-connected \( R \)-algebraic group.

We establish a connection between the o-minimal universal covering groups of an abelian definably compact definably connected group definable in \( R \) and of the definable identity component \( H (R)^0 \) of the group of \( R \)-points \( H (R) \) of some Zariski-connected \( R \)-algebraic group \( H \). More precisely we show Theorem 5.6.2.

**Theorem 5.6.2.** The o-minimal universal covering group of an abelian definably compact definably connected group definable in a sufficiently saturated real closed field \( R \) is, up to locally definable isomorphisms, an open locally definable subgroup of the o-minimal universal covering group of the definable identity component \( H (R)^0 \) of the group of \( R \)-points \( H (R) \) of some Zariski-connected \( R \)-algebraic group \( H \).

We were inspired by the closed relation between groups definable in a field \( F \) and \( F \)-algebraic groups. Given an \( F \)-algebraic group \( H \), the group of \( F \)-points \( H (F) \) is a definable group in \( F \). When \( F \) is an algebraically closed field, every definable group in \( F \) is \( F \)-definably isomorphic, as an \( F \)-definable group, to some \( F \)-algebraic group ([7, 47]); this fact is a version of a theorem of Weil that asserts that any \( F \)-algebraic group can be recovered from birational data [49]. However, when \( F \) is real closed, there are semialgebraic groups over \( F \) that are not \( F \)-definably isomorphic to \( H (F) \) for any \( F \)-algebraic group \( H \) (e.g., consider the group \(([0,1] \subseteq F, +_{\text{mod}1})\)).

Hrushovski and Pillay formulated in [21, 22] a relationship between a semialgebraic group \( G \) over a real closed field \( R \) and the set of \( R \)-points \( H (R) \) of some \( R \)-algebraic group \( H \). It roughly states that although the group operation of a semialgebraic group is given by a semialgebraic function, it is locally given by a rational function. More specifically, it assures the following.

**Fact 1.0.1.** [21, 22, Thm. A] Let \( G \) be a definably connected group definable in real closed field \( R = (R, <, +, 0, \cdot, 1) \). Then there are a Zariski-connected \( R \)-algebraic group \( H \), a definable neighborhood \( U \) of the identity of \( G \), and a definable homeomorphism \( f : U \to f (U) \subseteq H (R) \), where \( H (R) \) is the set of \( R \)-points of \( H \), such that \( x, y, xy \in U \) implies \( f (xy) = f (x) f (y) \).

Nevertheless, the neighborhood around the identity of \( G \) given by the above result of Hrushovski and Pillay could not give enough information about \( G \); for instance,
if $U$ is too small. Consider the following example, the group $([0,1) \subseteq \mathbb{R}, +_{\text{mod} 1})$ with addition modulo 1 is locally homomorphic to $(\mathbb{R}, +)$, where $(\mathbb{R}, <, +, 0, \cdot, 1)$ is a $\aleph_1$-saturated real closed field. More precisely, the definable bijection $f : [0, \beta) \cup (1 - \beta, 1) \to (-\beta, \beta)$ defined by $f(x) = x$ if $x \in [0, \beta)$, or $f(x) = x - 1$ if $x \in (1 - \beta, 1)$, where $0 < \beta < \frac{1}{n}$ for every $n \in \mathbb{N}$, is a local homomorphism between $([0,1) \subseteq \mathbb{R}, +_{\text{mod} 1})$ and $(\mathbb{R}, +)$, where by a local homomorphism between two groups we mean a map between some neighborhoods of their identities that acts as a group homomorphism inside its domain (see Def. 4.2.1). But $U = [0, \beta) \cup (1 - \beta, 1)$ cannot cover $G$ with finitely many group translates, and even the subgroup $\langle U \rangle$ generated by $U$ cannot say anything about the torsion of $G$. This situation led us to question: which conditions should $G$ have in order to obtain a “good and useful” local homomorphism between $G$ and $H(\mathbb{R})$? And how could such a map be used to characterize the group $G$?

Fortunately, the definably compactness of $G$ allows us to obtain a “good and useful” local homomorphism between $G$ and $H(\mathbb{R})$, and this map is good in the sense that its domain is a generic definable set in $G$. This is precisely Theorem 4.3.1.

**Theorem 4.3.1.** Let $G$ be a definably compact definably connected group definable in a sufficiently saturated real closed field $\mathbb{R}$. Then there are

(i) a Zariski-connected $\mathbb{R}$-algebraic group $H$ such that $\dim(G) = \dim(H(\mathbb{R})) = \dim(H),$

(ii) a definable $X \subseteq G$ such that $G^{00} \subseteq X,$

(iii) a definable homeomorphism $\phi : X \subseteq G \to \phi(X) \subseteq H(\mathbb{R})$ such that $\phi$ and $\phi^{-1}$ are local homomorphisms,

where by $G^{00}$ we denote the smallest type-definable subgroup of $G$ of index $< \kappa$ (see Def. 2.2.3 and Notation 2.2.4).

To prove the above result we introduce the notion of group-generic point in Chapter 3. An element $a$ of a group $G$ definable over $A \subseteq M$ is called a group-generic point of $G$ over $A$ if every $A$-definable $X \subseteq G$ with $a \in X$ is generic in $G$ (namely, $X$ covers $G$ by finitely many group translates), where $M = (M, <, \ldots)$ is a sufficiently saturated o-minimal structure. We show the existence of group-generic points in definably compact groups definable in a sufficiently saturated o-minimal expansion of a real closed field (Prop. 3.3.4) as well as establishing some properties of generic,
group-generic points, and generic sets. With these tools we adapt the proof of [21, Prop. 3.1] to obtain a strong version of the group configuration result for definably compact groups (Prop. 4.1.2), which is one of the main ingredients for the proof of Theorem 4.3.1.

As a consequence of our work, we also present a characterization of the o-minimal universal covering groups of the one-dimensional semialgebraically connected semialgebraic groups over a sufficiently saturated real closed field (Thm. 6.0.3).

**Theorem 6.0.3.** Let \( G \) be a 1-dimensional definably connected group definable in a sufficiently saturated real closed field \( \mathcal{R} = (R, <, +, 0, \cdot, 1) \). Then the o-minimal universal covering group \( \tilde{G} \) of \( G \) is locally \( \mathcal{R} \)-definably isomorphic, as a locally definable group, to a connected locally definable group \( W \) that is either the o-minimal universal covering group \( \tilde{H}(R)^0 \) of \( H(R)^0 \) or an open subgroup of \( H(R)^{00} \) for some 1-dimensional Zariski-connected \( R \)-algebraic group \( H \).

We recall that the \( R \)-points of a 1-dimensional Zariski-connected \( R \)-algebraic group are fully classified (see Fact 2.5.2).

**Remark 1.0.2.** Connectedness. Throughout the dissertation we will use different notions of connectedness that we clarify now.

(i) Let \( H \) be an algebraic group, we say that \( H \) is Zariski-connected if \( H \) is a connected space with the Zariski topology (as e.g., in Def. 2.4.1, Fact 2.4.2, and Theorems 4.3.1 and 5.6.2).

(ii) Let \( K \) be an algebraic closed field. Let \( H \) be a group definable in \( K \). \( H \) is called definably connected if \( H \) has no proper \((K-)\)definable subgroup of finite index (as in Fact 2.4.2 and Prop. 4.1.2).

(iii) Let \( G \) be a definable group in an o-minimal structure. As we defined in the first part of the Introduction, \( G \) is called definably connected if \( G \) has no nonempty proper definable t-clopen subset, where the t-topology on \( G \) is given by [35], (as in Theorems 4.3.1 and 5.6.2). In case \( G \) is a definable group in a real closed field (i.e., a semialgebraic group), we also say that \( G \) is semialgebraically connected to say that \( G \) is definably connected.

(iv) Let \( U \) be a \( \bigvee \)-definable group in a sufficiently saturated o-minimal structure. \( U \) is called \((\tau-)\)connected if \( U \) has no nonempty proper \( \tau \)-clopen subset such
that its intersection with any definable subset of $U$ is definable, where the $\tau$ topology on $U$ is given by [32, Prop. 2.2], (as in Def. 2.2.8, Fact 2.3.1, and results in Chapter 5). Note that if $U$ is a definable group, then the $t$-topology that exist on $U$ given by [35] agrees with the $\tau$-topology on $U$ as a $\lor$-definable group, thus, in this case, $U$ is connected as a $\lor$-definable group if and only if $U$ is definably connected as a definable group.

(v) We also have a notion of connected ld-space, see Def. 5.1.4. Note that by [2, Theorem 3.9], every $\lor$-definable group $U$ with its $\tau$ topology is an ld-space of finite dimension, and its notions of connectedness as a $\lor$-definable group and as an ld-space coincide.

### 1.1 The structure of the thesis

Chapter 2 contains some basic background used throughout the dissertation. In Chapter 2, Section 2.1, we define real closed fields and their definable objects: the semialgebraic sets. Some basic definitions, topological notions and properties of $\lor$-definable groups are presented in Section 2.2. In Section 2.3 we discuss some results about one-dimensional definably connected groups definable in an o-minimal structure, among them we note that such groups are either torsion free or definably compact. The purpose of the last two sections (Sections 2.4 and 2.5) is to review some notions, examples, and facts about algebraic groups and their interplay with semialgebraic groups.

Group-generic points, generic points, and generic sets are introduced and studied in Chapter 3. We define group-generic points for $\lor$-definable groups and show their existence in definably compact groups definable in a sufficiently saturated o-minimal expansion of a real closed field (Prop. 3.3.4) as well as establishing their connection with generic points and generic sets. These features are applied in the proof of one of the key findings of this work: Theorem 4.3.1. The last section of this chapter (Section 3.4) contains some results on generic definable sets in the product group $G \times G$ for a definably connected definably compact group $G$ definable in a sufficiently saturated o-minimal expansion of a real closed field. One of those results relates the genericity of a definable subset of $G \times G$ with the set of points whose fiber is generic (Lemma 3.4.3), and the other fact assures the existence of a definable generic box inside a generic definable subset of $G \times G$ (Lemma 3.4.9).
In Chapter 4 we prove one of the main results of this work: Theorem 4.3.1 on the existence of a definable injective local homomorphism with generic definable domain between a given definably compact definably connected group definable in a sufficiently saturated real closed field $R$ and the set of $R$-points of some $R$-algebraic group. For its proof we show a group configuration proposition for definably compact groups in Section 4.1, and provide some notions and remarks on local homomorphisms in Section 4.2.

Chapter 5 is devoted to the study of locally definable covering homomorphisms of locally definable groups in a sufficiently saturated o-minimal expansion of a real closed field. In Section 5.1 we examine the locally definable spaces, ld-spaces, and their ld-covering maps as are defined in [2]. The $\bigvee$-definable groups are examples of such spaces. We show that any ld-covering map between ld-spaces is closed for definable subspaces (Prop. 5.1.13). The o-minimal universal covering homomorphism of a connected locally definable group is introduced and studied in Section 5.2. Sections 5.3 and 5.4 investigate the connection between abelian definably generated groups (see the definition of definably generated group in Def. 2.2.1), existence of generic definable sets, convex sets, and covers of definable groups. We prove in Proposition 5.3.7 the existence of a convex set inside a definable generic subset of an abelian $\bigvee$-definable group $U$ with $U^{00}$. This is a crucial fact in the construction of a well defined covering map in Theorem 5.5.1.

In the last two sections of Chapter 5 we present another of the central results of this work: the extension of a definable local homomorphism from a torsion free abelian definably generated group under some additional assumptions (Thm. 5.5.1), and a relationship between the o-minimal universal covering groups of some connected abelian locally definable groups that are locally homomorphic (Theorem 5.6.1). Finally, by means of Theorems 5.6.1 and 4.3.1, we get Theorem 5.6.2.

In Chapter 6 we combine some properties of one-dimensional definably connected groups definable in an o-minimal structure presented in Section 2.3 with the study of locally definable covering homomorphisms to carry out a description of the o-minimal universal covering groups of the one-dimensional definably connected groups definable in a sufficiently saturated real closed field (see Thm. 6.0.3).

Appendix A has the goal to provide the reader the details of the proof of Proposition 3.1 in [21] that where cited throughout the proof of Proposition 4.1.2.

**Notation.** Throughout this dissertation, given a (model-theoretic) structure $\mathcal{M}$, definable means $\mathcal{M}$-definable possibly with parameters. Our notation and any undefined
term that we use from model theory, topology, or algebraic geometry are generally standard. For a group $G$ whose group operation is written multiplicatively, we use the following notation $\prod_{n} X = X \cdot \ldots \cdot X$, and $X^n = \{x^n : x \in X\}$ for any $n \in \mathbb{N}$. 
Chapter 2

Preliminaries

In this chapter we cover some basic background on real closed fields, semialgebraic groups, $\mathcal{V}$-definable groups, and algebraic groups for later use.

In the first two sections we define some of the central objects of this work: the real closed fields and their definable objects: the semialgebraic sets. Also, we introduce the notion of $\mathcal{V}$-definable group and point out one of the key facts of these groups: the existence of a topology making the group into a topological group ([32, Prop. 2.2]), among other results.

In Section 2.3 we review some of the known properties of one-dimensional groups definable in an o-minimal structure that are applied later in Chapter 6. Among them we note the two mutually exclusive properties of the definably connected groups definable in an o-minimal structure: they are either torsion free or definably compact (Fact 2.3.1). We also state a relationship between some definable neighborhood of the identity element of one of these groups $G$ and its type-definable group $G^{00}$ (Corollary 2.3.5).

The last two sections (2.4 and 2.5) are devoted to stating some definitions, facts, and examples on algebraic groups as well as some of their connections with the semialgebraic groups.
2.1 Real closed fields and their definable sets

Definition 2.1.1. Let $\mathcal{R} = (R, <, +, 0, \cdot, 1)$ be an ordered field. $\mathcal{R}$ is called a real closed field if $\mathcal{R}$ is elementarily equivalent to the real ordered field $\mathbb{R}$; equivalently, if every positive element of $R$ is a square and every polynomial of odd degree over $R$ has a root in $R$.

Examples 2.1.2. Some real closed fields are the real ordered field $\mathbb{R}$, the real algebraic numbers $\mathbb{R}_{\text{alg}}$, the hyperreal numbers $\mathbb{R}^*$, which is a $\aleph_1$-saturated expansion of $\mathbb{R}$ with infinite and infinitesimal elements, the Puiseux series $\mathbb{R}(\![X]\!)$ with coefficients over a real closed field $\mathbb{R}$ in the $X$ indeterminate (a series in this field is of the form $\sum_{i=k}^{\infty} a_i X^{\frac{n}{n}}$ for $k \in \mathbb{Z}$, $n \in \mathbb{N} \setminus \{0\}$, [5]), among other examples.

Definition 2.1.3. Let $\mathcal{R} = (R, <, +, 0, \cdot, 1)$ be an ordered field. A set $X \subseteq R^n$ is called semialgebraic over $R$ if $X$ is a finite Boolean combination of sets of solutions of $p(x) = 0$ and $q(x) > 0$ with $p(x), q(x)$ polynomials over $R$; namely, $X \subseteq R^n$ is a finite union of sets of the form

$$\{x \in R^n : p_1(x) = 0, \ldots, p_l(x) = 0, q_1(x) > 0, \ldots, q_s(x) > 0\}$$

for some polynomials $p_i, q_j \in R[x]$, $1 \leq i \leq l$, $1 \leq j \leq s$.

We say that a map between semialgebraic sets is a semialgebraic map (over $R$) if its graph is a semialgebraic set (over $R$).

A group $(G, \cdot, i)$ is called a semialgebraic group (over $R$) if $G$, its group operation $\cdot : G \times G \to G : (g_1, g_2) \mapsto g_1 \cdot g_2$, and its inversion map $i : G \to G : g \mapsto g^{-1}$ are semialgebraic (over $R$).

One of the properties of a real closed field is that it has quantifier elimination.

Fact 2.1.4. (Tarski-Seidenberg) Let $\mathcal{R} = (R, <, +, 0, \cdot, 1)$ be a real closed field, then every $\mathcal{R}$-definable subset of $R^n$ is semialgebraic.

Then all the definable groups in a real closed field $\mathcal{R}$ are the semialgebraic groups over $\mathcal{R}$. We will see some examples of these groups in Examples 2.5.3.

Definition 2.1.5. Let $\mathcal{M} = (M, <, \ldots)$ be a structure such that $<$ is a dense linear ordering without endpoints. $\mathcal{M}$ is o-minimal if every definable subset of $M$ is a finite union of points and intervals with endpoints in $M \cup \{\pm \infty\}$.
Other properties of the theory of the real closed fields are that it is decidable, complete, and o-minimal (for more on o-minimality see e.g. [48]).

2.2 $\bigvee$-definable groups

Recall that by a sufficiently saturated structure we mean a $\kappa$-saturated structure for some sufficiently large cardinal $\kappa$.

In what follows, we introduce the definition of $\bigvee$-definable groups and locally definable groups. These groups have been used as a tool for the study of the definable groups in o-minimal structures, and also have appeared in connection with the definable groups. E.g., the work of Peterzil and Starchenko in [32] uses the group of definable homomorphisms between two abelian definable groups (which is a locally definable group) in the proof of the fact that every definably compact solvable group is abelian by finite. Some interpretability problems have been also considered by Peterzil, Pillay, and Starchenko in [31] using locally definable groups to show that the group structure of a definable group that is not nilpotent by finite interprets a field. In this work we use the o-minimal universal covering group (see Definition 5.2.3), which is a locally definable group, as a tool to study groups definable in a real closed field.

The general theory of locally definable groups was introduced by Edmundo in [12]. Locally definable groups belong to a wider category: the locally definable spaces (see Sect. 5.1) established by Baro and Otero in [2].

**Definition 2.2.1.** Let $\mathcal{M} = (M, <, \ldots)$ be a sufficiently saturated structure. A $\bigvee$-definable group is a group $(\mathcal{U}, \cdot)$ whose universe is a union $\mathcal{U} = \bigcup_{i \in I} Z_i$ of $\mathcal{M}$-definable subsets of $M^n$ for some fixed $n$, all defined over $A \subseteq M$ with $|A| < \kappa$ such that for every $i, j \in I$

(i) there is $k \in I$ such that $Z_i \cup Z_j \subseteq Z_k$ (i.e., the union is directed), and

(ii) the group operation $\cdot |_{Z_i \times Z_j}$ and group inverse $(\cdot)^{-1} |_{Z_i}$ are $\mathcal{M}$-definable maps into $M^n$.

We say that $(\mathcal{U}, \cdot)$ is locally $\mathcal{M}$-definable if $|I|$ is countable.
A subgroup \( W \) of a \( \lor \)-definable group \( U \) is called \textit{definably generated} if there is some definable subset \( X \subseteq U \) such that the group \( \langle X \rangle \) generated by \( X \) is equal to \( W \); therefore, \( W \) is in particular a locally definable group.

A map between \( \lor \)-definable (locally definable) groups is called \( \lor \)-definable (locally \( \mathcal{M} \)-definable) if its restriction to any \( \mathcal{M} \)-definable set is a \( \mathcal{M} \)-definable map.

**Definition 2.2.2.** Let \( \mathcal{M} = (M, <, \ldots) \) be an \( \omega \)-minimal structure. Let \( X \subseteq M^m \) \( \mathcal{M} \)-definable, the \textit{(geometric) dimension of} \( X \), \( \dim(X) \), is the maximal \( n \leq m \) such that the projection of \( X \) onto \( n \) coordinates contains an open set of \( M^n \), where we use the order topology on \( M \) and the product topology on \( M^l \) for any \( l \in \mathbb{N} \).

For a \( \lor \)-definable group \( U = \bigcup_{i \in I} Z_i \), we define \( \dim(U) = \max \{ \dim(Z_i) : i \in I \} \).

**Definition 2.2.3.** Let \( \mathcal{M} \) be a sufficiently saturated structure. A \textit{type-definable set} in \( \mathcal{M} \) is a subset of \( M^n \) that is the intersection of less than \( \kappa \)-many definable sets. If \( A \subseteq M \), we say that a set is type-definable over \( A \) if the set can be defined by a set of formulas with parameters from \( A \).

**Notation 2.2.4.** For a \( \lor \)-definable group \( U \), we denote by \( U^{00} \) the smallest, if such exists, type-definable subgroup of \( U \) of index smaller than \( \kappa \). Recall that if \( U \) is a definable group in a NIP structure, then \( U^{00} \) exists ([43]), but if \( U \) is a \( \lor \)-definable group, \( U^{00} \) may not exist (see Example 2.2.6(iii)). Facts 5.3.1 and 5.3.3 summarize some results from [4, 17] about the existence of \( U^{00} \) for a connected abelian definably generated group \( U \) in a sufficiently saturated \( \omega \)-minimal expansion of an ordered group.

We recall that given a \( \lor \)-definable group \( U \) such that \( U^{00} \) exists we can endow the quotient group \( U/U^{00} \) with a topology as is outlined below.

**Fact 2.2.5.** [19, Lemma 7.5] Assume \( U = \bigcup_{i \in I} Z_i \) is a \( \lor \)-definable group in a sufficiently saturated model and that \( U^{00} \) exists. Let \( \pi : U \to U/U^{00} \) be the canonical projection map and set \( C \subseteq U/U^{00} \) to be closed if and only if for every \( i \in I \), \( \pi^{-1}(C) \cap Z_i \) is type-definable. Then these closed sets generate a locally compact topology on \( U/U^{00} \) making it into a Hausdorff topological group. We call this topology on \( U/U^{00} \) the logic topology.

In this work we are mostly interested in locally definable groups and definable groups.
Examples 2.2.6. Let $\mathcal{M}$ be a sufficiently saturated structure.

(i) Let $G$ be a $\mathcal{M}$-definable group, and $X \subseteq G$ definable containing the identity element of $G$. Then the subgroup $\mathcal{U} = \langle X \rangle = \bigcup_{n \in \mathbb{N}} X \cdot X^{-1}$ of $G$ generated by $X$ is a locally $\mathcal{M}$-definable group. Then, in particular, every countable group is a locally definable group in any structure as well as the commutator subgroup $[G, G]$ of a $\mathcal{M}$-definable group $G$.

(ii) If in addition $\mathcal{M}$ is an o-minimal structure, the o-minimal universal covering group (see Definition 5.2.3) of a connected locally $\mathcal{M}$-definable group exists, by [14, Thm. 3.11], and is a locally $\mathcal{M}$-definable group.

(iii) ([17]) Let $(G, <, +)$ be a sufficiently saturated ordered divisible abelian group, and in it take an infinite increasing sequence of elements $0 < a_1 < a_2 < \cdots$ such that $na_i < a_{i+1}$ for every $n \in \mathbb{N}$. The subgroup $\mathcal{U} = \bigcup_i (-a_i, a_i)$ of $G$ is a $\vee$-definable group. This group has the distinction of having no $\mathcal{U}^{(0)}$, and is not definably generated.

(iv) If $\mathcal{M}$ is an o-minimal expansion of an ordered field $(R, <, +, 0, \cdot, 1)$ and $\alpha \in R^{>0}$, the map

$$f : \bigcup_{n \in \mathbb{N}} [-n\alpha, n\alpha] \to [0, \alpha) \quad x \mapsto x \mod \alpha$$

is a locally $\mathcal{M}$-definable map, which is also a group homomorphism between $(\bigcup_{n \in \mathbb{N}} [-n\alpha, n\alpha], +) \subseteq (R, +)$ and $([0, \alpha), +_{\mod \alpha})$. In fact, $f$ is the o-minimal universal covering homomorphism of the $\mathcal{M}$-definable group $([0, \alpha) \subseteq M, +_{\mod \alpha})$ with addition modulo $\alpha$ (see Def. 5.2.3).

Some additional examples of definable groups of dimension one are in Examples 2.5.3.

Any $\vee$-definable group in a sufficiently saturated o-minimal structure $\mathcal{M}$ can be endowed with a topology $\tau$ making it into a topological group. This fact was a generalization by Peterzil and Starchenko [32] of the known result for the definable groups by Pillay [35]. In case $\mathcal{M}$ expands the reals, that topology makes any definable group into a real Lie group. Moreover, any $\vee$-definable homomorphism between two $\vee$-definable groups is continuous with respect to their $\tau$ topologies [32, Lemma 2.8], and any $\vee$-definable subgroup $\mathcal{W}$ of a $\vee$-definable group $\mathcal{U}$ is $\tau$-closed in $\mathcal{W}$, and if $\mathcal{W}$ and $\mathcal{U}$ have the same dimension, then $\mathcal{W}$ is also $\tau$-open in $\mathcal{W}$ [32, Lemma 2.6].
For what follows, given a $\bigvee$-definable group $U$ defined over $A \subseteq M$ in a sufficiently saturated o-minimal structure, an element $a \in U$ is a generic point of $U$ over $A$ if its (acl-)dimension $\dim (a/A)$ is equal to $\dim (U)$.

**Fact 2.2.7.** [32, Prop. 2.2] Assume $\mathcal{M}$ is a sufficiently saturated o-minimal structure. Let $U \subseteq M^n$ be a $\bigvee$-definable group. Then there is a uniformly definable family $\{V_a : a \in S\}$ of subsets of $U$ containing the identity element $e_U$ and a topology $\tau$ on $U$ such that $\{V_a : a \in S\}$ is a basis for the $\tau$-open neighbourhoods of $e_U$ and $U$ is a topological group. Moreover, every generic $h \in U$ has an open neighborhood $O \subseteq M^n$ such that $O \cap U$ is $\tau$-open and the topology that $O \cap U$ inherits from $\tau$ agrees with the topology it inherits from $M^n$, and the topology $\tau$ is the unique one with the above properties.

**Definition 2.2.8.** Assume $\mathcal{M}$ is a sufficiently saturated o-minimal structure. Let $U$ be a $\bigvee$-definable group with its $\tau$-topology.

(i) $U$ is ($\tau$-)connected if $U$ has no nonempty proper ($\tau$-)clopen subset such that its intersection with any definable subset of $U$ is definable.

(ii) $U$ is ($\tau$-)definably compact if every definable path $\gamma : (0,1) \to U$ has limits points in $U$ (where the limits are taken with respect to the $\tau$-topology on $U$).

Note that if $U$ is a definable group, then the t-topology that exist on $U$ given by [35] agrees with the $\tau$-topology on $U$ as a $\bigvee$-definable group, and thus the above notions of connectedness and compactness agree with those given in the Introduction for definable groups.

$SO(2,R)$, $([0,1],+_\text{mod}1)$, $([1,a],'_\text{mod}a)$, and $E(R)$ are examples of definably compact definably connected groups definable in a real closed field $R$.

### 2.3 Some results of one-dimensional groups definable in o-minimal structures

Among some of the research on one-dimensional groups definable in an o-minimal structure $\mathcal{M} = (M, <, \ldots)$, we point out the work of Razenj [39] where he classified the one-dimensional $\mathcal{M}$-definable groups, but up to abstract group isomorphisms.
There is also a description of such groups given by Strzebonski in [45] where he dealt with definably connected groups $G \subseteq M^n$ whose group operation is continuous with respect to the product topology of $M^n$ given by the order topology on $(M, <)$. In case the structure is the ordered real field $(\mathbb{R}, <, +, 0, \cdot, 1)$, Madden and Stanton in [26] provided a full characterization of the 1-dimensional semialgebraic groups over $\mathbb{R}$ up to semialgebraic group isomorphisms; in this setting the definable groups correspond to the Nash groups (semialgebraic and analytic groups over $\mathbb{R}$).

Our work also presents a contribution to the study of the one-dimensional groups definable in a real closed field.

*From now until the end of this section, we assume that $\mathcal{M}$ is a sufficiently saturated o-minimal expansion of a real closed field $(\mathbb{R}, <, +, 0, \cdot, 1)$. And by definable we always mean $\mathcal{M}$-definable.*

In what follows, we will present some properties of 1-dimensional groups definable in $\mathcal{M}$.

Peterzil and Steinhorn proved in [34, Theorem 1.2] that every one-dimensional definably connected definable group satisfies a dichotomy: either the group is torsion free or definably compact. However, there are one-dimensional connected locally definable groups which are both torsion free and definably compact; e.g., consider $\bigcup_{n \in \mathbb{N}} (-n, n) \subseteq (\mathbb{R}, +)$. And by the next fact of Edmundo every one-dimensional connected locally definable group that is not torsion free must be definable and definably compact.

**Fact 2.3.1.** [12, Corollary 8.3] Let $U$ be a connected locally definable group of dimension one. Then $U$ is abelian, and either $U$ is torsion free and locally definably totally ordered or $U$ is a definably compact definable group.

From the above and the next result of Miller and Starchenko, the 1-dimensional torsion free definably connected groups definable in a real closed field are fully determined.

**Fact 2.3.2.** [27] If $\mathcal{M}$ is in addition a polynomially bounded structure, then every definable one-dimensional ordered group is definably isomorphic, as definable groups, to the additive group $(\mathbb{R}, +)$ or the multiplicative group of positive elements $(\mathbb{R}^>0, \cdot)$.

Where by a *polynomially bounded structure* we mean a structure where any unary definable map is bounded at infinity by some power function, i.e., by a definable
endomorphism of \((R^>0,)\). Recall that any real closed field is polynomially bounded ([48]).

Thus, from Facts 2.3.1 and 2.3.2 we conclude that every one-dimensional torsion free group definable in \(R\) is \(R\)-definably isomorphic to \((R,+)\) or \((R^>0,)\). Moreover, it is conjectured that every abelian torsion free group definable in a polynomially bounded structure is a direct sum of definable one-dimensional subgroups [33, Conjecture 1].

2.3.1 An orientation on one-dimensional definably compact semialgebraic groups

Here we recall some properties and notation of 1-dimensional definably compact groups from [37, 39]. We use them in the proof of Proposition 2.3.4.

Fact 2.3.3. [37, 39] Let \(G\) be a 1-dimensional definably compact definably connected definable group with identity \(e_G\).

(i) \(G\) can be endowed with a definable “circular” orientation \(R\) on \(G\) satisfying \(R(x,y,z)\) implies \(x, y, z\) are pairwise distinct, \(R(x,y,z) \to R(y,z,x)\), \(R(x,y,z) \to R(z^{-1}, y^{-1}, x^{-1})\) and \(R(x,y,z)\) implies \(R(xw,yw,zw)\) for any \(w\).

(ii) For every \(x \in G\), \(R(x,y,z)\) defines a dense linear ordering without endpoints on \(G \setminus \{x\}\). Denote this ordering on \(G \setminus \{x\}\) by \(<_x\).

(iii) The topology on \(G\) as a definable group [35] agrees with the topology on \(G\) given by the \(<_x\)'s, and every definable subset of \(G \setminus \{x\}\) is a finite union of points and \(<_x\)-intervals.

(iv) If \(a_0 \in G \setminus \{e_G\}\) and \(a_0^2 = e_G\), then \(e_G <_{a_0} x \iff x^{-1} <_x e_G\) for every \(x \in G \setminus \{a_0\}\). Let \(<\) denote \(<_{a_0}\) on \(G \setminus \{a_0\}\). Then, by [37], \(G^{00} = \bigcap_{a \in T(G), e_G < a} (a^{-1}, a)\), where \(T(G)\) is the torsion subgroup of \(G\).

Proposition 2.3.4. Let \(G\) be a 1-dimensional definably compact definably connected definable group with identity \(e_G\). Let \(V \subseteq G\) be a connected symmetric definable neighborhood of \(e_G\) without the nontrivial element \(a_0\) of order two of \(G\). Then either \(V \subseteq G^{00}\) or \(G^{00} \subseteq V\).
Proof. Let $<$ denote the dense linear ordering $<_a$ on $G\backslash\{a_0\}$ defined by the “circular” orientation on $G$. Then there is $v \in G$ such that the interior of $V$ is equal to $(v^{-1}, v)$. We have two cases. If $V$ has no nontrivial torsion point, then for any $a \in T(G)$ and $e_G < a$, we have that $v < a$, thus, $V \subseteq \bigcap_{a \in T(G), e_G < a} (a^{-1}, a) = G^{00}$. Otherwise, $V$ has a nontrivial torsion point; i.e., there is $a \in T(G), e_G < a$ such that $v^{-1}, a) \subseteq V$, so $G^{00} \subseteq V$.

\[\square\]

Corollary 2.3.5. Let $G$ be a 1-dimensional definably connected definable group with identity $e_G$. Let $V \subseteq G$ be a connected symmetric definable neighborhood of $e_G$ without the nontrivial element $a_0$ of order two of $G$. Then either $V \subseteq G^{00}$ or $G^{00} \subseteq V$.

Proof. By [34, Theorem 1.2], $G$ is either torsion free or definably compact. In the torsion free case, $G = G^{00}$ (see proof of [37, Prop. 3.5]), so clearly $V \subseteq G^{00}$. Finally, the definably compact case is given by Proposition 2.3.4.

\[\square\]

2.4 On algebraic groups

In this section we introduce some basic definitions and facts about algebraic groups as well as some classical examples. For more on algebraic groups see [6, 9, 40, 41, 42, 50].

Definition 2.4.1. Let $K$ be an algebraically closed field, and $k \subseteq K$ a subfield of $K$. An algebraic group defined over $k$, or $k$-algebraic group, is a group $(H, \cdot, i)$ such that $H$ is an algebraic variety defined over $k$ and the group operation $\cdot : H \times H \to H : (h_1, h_2) \mapsto h_1 \cdot h_2$ and group inverse $i : H \to H : h \mapsto h^{-1}$ are morphisms defined over $k$. We say that a $k$-algebraic group $H$ is Zariski-connected if $H$ is a connected space with the Zariski topology; this is equivalent to say that $H$ has no proper Zariski open subgroup.

For a group $G$ definable in an algebraically closed field $K$, we say that $G$ is definably connected if there is no proper definable subgroup of $G$ of finite index.
By the next fact, every group definable in an algebraically closed field $K$ is $K$-
definably isomorphic to some algebraic group defined over $K$. This fact is a special

case of the Theorem of Weil that asserts that any $K$-algebraic group can be recovered

from birational data [49], and some of its proofs were given by van den Dries [47]

and Hrushovski [7].

**Fact 2.4.2.** [7, 47] Let $K$ be an algebraically closed field, and $k \subseteq K$ a subfield of

$K$. Let $G$ be a definably connected group definable in $K$. Then there is a Zariski-

connected $K$-algebraic group $H$ and a $K$-definable isomorphism $f$ between $G$ and $H$.

If moreover $\text{char } (K) = 0$ and $G$ is definable with parameters in $k$, then both $H$ and $f$

can be chosen to be $k$-definable.

**Examples 2.4.3.** Some examples of algebraic groups over an algebraically closed

field $K$ are the following.

(i) The additive group $K_a$ of the field $K$, and its multiplicative group $K_m$.

(ii) Any diagonalizable group; i.e, a Zariski-closed subgroup of the group $(K_m)^n$ of

diagonal matrices of size $n \times n$. Any group $K$-isomorphic to a diagonalizable

group is called a torus. Note that if $R$ is an ordered field and $K = R \left( \sqrt{-1} \right)$ is

its algebraic closure, then the special orthogonal group

$$SO(2, K) = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a^2 + b^2 = 1, a, b \in K \right\}$$

is a torus because $\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mapsto a + \sqrt{-1}b$ is a $K$-isomorphism between

$SO(2, K)$ and $K_m$.

(iii) $GL(K, n)$ the group of the $n \times n$ invertible matrices over $K$ with group operation

given by matrix multiplication.

(iv) Any Zariski closed subgroup of $GL(K, n)$; namely, a linear algebraic group. E.g.,

the group $SL(n, K)$ of all $n \times n$ matrices of determinant 1.

(v) An elliptic curve $E$; i.e, a projective algebraic group that is the set of solutions

in the projective plane $\mathbb{P}^2(K)$ of an equation of the form

$$y^2z = x^3 + axz^2 + bz^2$$

for some $a, b \in K$. 

25
(vi) Any abelian variety, which is by definition an algebraic projective group. Abelian varieties of dimension one are precisely the elliptic curves.

**Remark 2.4.4.** From the known classification of Zariski-connected one-dimensional algebraic groups over an algebraically closed field $K$ (see [44]), we have that, up to $K$-isomorphisms of $K$-algebraic groups, the only Zariski-connected $K$-algebraic groups of dimension one are the following:

(i) The additive group $K_a$.

(ii) The multiplicative group $K_m$.

(iii) An elliptic curve $E$ over $K$.

### 2.5 Algebraic and semialgebraic groups

Let $K$ be an algebraically closed field, and $k \subseteq K$ a subfield of $K$. Note that if $H$ is a $k$-algebraic group, the group of the $k$-rational points $H(k)$ of $H$ is a group definable in $k$. Therefore, for every $R$-algebraic group $H$, $H(R)$ is a definable group in $R$, where $R$ is a real closed field.

In case $H$ is an abelian linear algebraic group defined over $R$, the definable identity component $H(R)^0$ of $H(R)$ is fully characterized.

**Fact 2.5.1.** [28, Fact 3.1] Let $H$ be a commutative linear algebraic group defined over a real closed field $(R,<,+,0,\cdot,1)$. Then the definable identity component $H(R)^0$ of $H(R)$ is definably isomorphic, as semialgebraic groups, to a group of the form $SO(2,R)^m \times (R^{>0},\cdot)^l \times (R,\cdot)^s$ for some $m,l,s \in \mathbb{N}$.

As a consequence of Remark 2.4.4 and Fact 2.5.1 we have the next fact.

**Fact 2.5.2.** Let $R = (R,<,+,0,\cdot,1)$ be a real closed field. Let $H$ be a 1-dimensional Zariski-connected algebraic group defined over $R$. Then the definable identity component $H(R)^0$ of $H(R)$ is definably isomorphic, as definable groups, to one of the following groups

(i) $(R,+)$
(ii) \((R^{>0}, \cdot)\)

(iii) \(SO (2, R)\)

(iv) \(E (R)^0\) for some elliptic curve \(E\) defined over \(R\).

Note that if \(H (R)^0\) is definably compact, then \(H (R)^0\) is definably isomorphic to either \(SO (2, R)\) or \(E (R)^0\).

Proof. By Remark 2.4.4, \(H\) is either a linear algebraic group or an elliptic curve \(E\) over \(K\), so suppose that \(H\) is a linear \(R\)-algebraic group. By Fact 2.5.1, \(H (R)^0\) is either \((R, +), (R^{>0}, \cdot)\), or \(SO (2, R)\).

\(\square\)

It should be noted that although the group \(H (R)\) of the \(R\)-rational points of every \(R\)-algebraic group \(H\) is a semialgebraic group over \(R\), there are semialgebraic groups over \(R\) that are not \(R\)-definably isomorphic to \(H (R)\) for any \(R\)-algebraic group \(H\), see Example 2.5.3(ii).

### 2.5.1 Examples of one-dimensional semialgebraic groups

#### Examples 2.5.3

Let \(\mathcal{R} = (R, <, +, 0, \cdot, 1)\) be a real closed field. The following are some examples of 1-dimensional definably connected groups definable in \(\mathcal{R}\).

(i) Let \(H\) be a 1-dimensional Zariski-connected \(R\)-algebraic group, then \(H (R)\) is a \(\mathcal{R}\)-definable group, whose definable identity component is determined by Fact 2.5.2.

(ii) Let \((\mathcal{W}, \oplus)\) be a 1-dimensional locally \(\mathcal{R}\)-definable group with a locally \(\mathcal{R}\)-definable ordering \(<_{\mathcal{W}}\) making \((\mathcal{W}, <_{\mathcal{W}}, \oplus)\) into an ordered group. Let \(e_{\mathcal{W}}\) be the identity element of \(\mathcal{W}\), and take an element \(a \in \mathcal{W} \setminus \{e_{\mathcal{W}}\}\) such that \(e_{\mathcal{W}} <_{\mathcal{W}} a\). We will define the addition modulo \(a \oplus_{moda}\) on the interval \([e_{\mathcal{W}}, a)\), where the addition is given by the group operation on \(\mathcal{W}\).
$$x \oplus_{\text{mod } a} y = \begin{cases} 
 x \oplus y & \text{if } x \oplus y <_{\mathcal{W}} a \\
 x \oplus y \ominus a & \text{if } a \leq_{\mathcal{W}} x \oplus y 
\end{cases}$$

Some particular examples of this group with addition modulo a fixed positive element can be defined when the group $\mathcal{W}$ is a locally $\mathcal{R}$-definable subgroup of $(\mathbb{R}, +)$, $(\mathbb{R}^{>0}, \cdot)$, $SO(2, \mathbb{R})^{00}$, or $E(\mathbb{R})^{00}$, for some elliptic curve $E$ defined over $\mathbb{R}$. 
Chapter 3

Group-generic points and generic sets

Assume throughout this chapter that $\mathcal{M} = (M, <, \ldots)$ is a sufficiently saturated $o$-minimal structure.

In this chapter we introduce the notion of group-generic point for $\bigvee$-definable groups (Def. 3.3.1). We study some of their properties and relationships with generic points. We point out the existence of group-generic points for definably compact groups when $\mathcal{M}$ is also an expansion of a real closed field (Prop. 3.3.4). The introduction and study of group-generic points is crucial in the proof of a group configuration proposition for definably compact groups (Prop. 4.1.2), and with this the proof of one of the main results of this work: Theorem 4.3.1.

3.1 Algebraic dimension and generic points

Recall that for $A \subseteq M$ and $b \in M$, $b$ is in the algebraic closure of $A$ ($b \in acl(A)$) if $b$ is an element of a finite $A$-definable set. And $b$ is in the definable closure of $A$ ($b \in dcl(A)$) if the singleton $\{b\}$ is $A$-definable. We can consider in these definitions of algebraic and definable closure finite tuples from $M$ instead of elements of $M$ with exactly the same definitions.

Note that since $\mathcal{M}$ is an ordered structure, $b \in acl(A) \leftrightarrow b \in dcl(A)$.

Fact 3.1.1. [38, Thm. 4.1](Exchange Lemma) If $A \subseteq M$ and $b, c \in M$ with $b \in$
\( \text{acl}(Ac) \setminus \text{acl}(A) \), then \( c \in \text{acl}(Ab) \).

From the Exchange Lemma we can define a model theoretic notion of dimension.

**Definition 3.1.2.** Let \( A \subseteq M \) and a tuple \( a \in M^n \). The \textit{(acl-)dimension of} \( a \) over \( A \), \( \dim(a/A) \), is the cardinality of any maximal \( A \)-algebraically independent subtuple of \( a \). If \( p \in S(A) \), then \( \dim(p) = \dim(a/A) \) for any tuple \( a \in M^n \) realising \( p \).

We recall some properties of this notion of dimension.

**Fact 3.1.3.** [35, Lemma 1.2] Let \( A, B \subseteq M \) and tuples \( a \in M^n \) and \( b \in M^m \).

(i) If \( A \subseteq B \), then \( \dim(a/A) \geq \dim(a/B) \).

(ii) \textit{(Additivity)} \( \dim(ab/A) = \dim(a/Ab) + \dim(b/A) \).

(iii) \textit{(Symmetry)} \( \dim(a/Ab) = \dim(a/A) \iff \dim(b/Aa) = \dim(b/A) \).

(iv) Let \( p \in S(A) \). If \( A \subseteq B \), there is \( q \in S(B) \) such that \( q \supseteq p \) and \( \dim(p) = \dim(q) \).

**Definition 3.1.4.** Let \( A \subseteq M \) and tuples \( a \in M^n \) and \( b \in M^m \).

(i) \( a \) is \textit{independent} from \( b \) over \( A \), denoted by \( a \perp_A b \), if \( \dim(a/A) = \dim(a/Ab) \).

(ii) Let \( X \subseteq M^n \) \( A \)-definable and \( a \in X \). \( a \) is a \textit{generic} point of \( X \) over \( A \) if \( \dim(a/A) = \dim(X) \).

Note, by [35, Lemma 1.4], that the \textit{(geometric) dimension} of an \( A \)-definable set \( X \subseteq M^n \) (see Definition 2.2.2) satisfies \( \dim(X) = \max \{ \dim(a/A) : a \in X \} \).

Since the structure \( M \) is sufficiently saturated, we have the following.

**Remark 3.1.5.** Let \( X \subseteq M^n \) definable over \( A \subseteq M \) with \( |A| < \kappa \). Then there is \( a \in X \) generic of \( X \) over \( A \).
3.2 Generic sets in $\bigvee$-definable groups

**Definition 3.2.1.** Let $\mathcal{U}$ be a $\bigvee$-definable group. A set $X \subseteq \mathcal{U}$ is *left* (right) generic in $\mathcal{U}$ if less than $\kappa$-many left (right) group translates of $X$ cover $\mathcal{U}$. $X$ is generic if it is both left and right generic, and $X$ is called $n$-generic if $n$-group translates of $X$ cover $\mathcal{U}$.

Thus, by saturation, a definable subset generic in a definable group covers the group in finitely many group translates.

**Fact 3.2.2.** [17, Fact 2.3(2)] Any left generic definable subset of a connected $\bigvee$-definable group $\mathcal{U}$ in an o-minimal structure generates $\mathcal{U}$.

Some examples of generic definable subsets of a definable group $G$ are the large subsets in $G$; namely, a definable set $Y \subseteq G$ such that $\dim (G \setminus Y) < \dim (G)$, this fact was proved by Pillay in [35, Lemma 2.4]. Also, note that if $\mathcal{M}$ admits the structure of an ordered field $(M, <, +, \cdot)$, then in the additive group $(M, +)$ a definable generic set $X \subseteq M$ is generic if and only if $M \setminus X$ is bounded in $M$ ([30, Remark 3.3]).

In case $G$ is a definably compact definably connected group, by [19, Prop. 4.2], for any definable set $X \subseteq G$, $X$ is left generic if and only if $X$ is right generic, so we just say generic.

**Fact 3.2.3.** [30, Thm. 3.7] Let $\mathcal{M}$ be a sufficiently saturated o-minimal expansion of a real closed field. Assume $G$ is a definably connected $\mathcal{M}$-definable group, and $X \subseteq G$ a definable set whose closure in $G$ is definably compact. If $X$ is not left generic in $G$, then $G \setminus X$ is right generic in $G$.

In the case of a $\bigvee$-definable group $\mathcal{U}$, there are some results about the existence of generic definable subsets. They are related with the existence of $\mathcal{U}^{00}$ and with the property of covering a definable group, see Facts 5.3.1 and 5.3.3.

**Remark 3.2.4.** Let $G$ be a group, $W \subseteq G$ a symmetric set. If there is $W' \subseteq W$ such that $WW \subseteq \bigcup_{i<\alpha} w_iW'$ for $\{w_i\}_{i<\alpha} \subseteq \langle W \rangle$ and some $\alpha < \aleph_1$, then $\langle W \rangle$ is contained in the union of strictly less than $\aleph_1$-many group translates of $W'$ by elements of $\langle W \rangle$; i.e., there are $\beta < \aleph_1$ and $\{x_i\}_{i<\beta} \subseteq \langle W \rangle$ such that $\langle W \rangle \subseteq \bigcup_{i<\beta} x_i W'$. 

31
Proof. As \( WW \subseteq \bigcup_{i<\alpha} w_i W' \) and \( W' \subseteq W \), then for every \( n \in \mathbb{N} \), \( \prod_n W \subseteq \bigcup_{i<\alpha_n} x_{i,n} W' \) for some \( \{x_{i,n}\}_{i<\alpha_n} \subseteq \langle W \rangle \) and some \( \alpha_n < \aleph_1 \). Thus, \( \langle W \rangle = \bigcup_{n \in \mathbb{N}} \prod_n W \subseteq \bigcup_{i<\beta} x_i W' \) for some \( \{x_i\}_{i<\beta} \subseteq \langle W \rangle \) and some \( \beta < \aleph_1 \). □

The above remark states in particular that if a definably generated subgroup \( \langle W \rangle \) of some \( \forall \)-definable group with \( W \) definable and symmetric contains a definable set \( W' \subseteq W \) that covers \( WW \) by finitely many group translates, then \( W' \) is generic in \( \langle W \rangle \) (if \( \kappa \geq \aleph_1 \)). We will apply this result in the proof of Proposition 5.4.1.

### 3.3 Group-generic points for \( \forall \)-definable groups

**Definition 3.3.1.** Let \( U \) be a \( \forall \)-definable group over \( A \subseteq M \) and \( a \in U \).

(i) \( a \) is a left (right) group-generic point of \( U \) over \( A \) if every \( A \)-definable \( X \subseteq U \) with \( a \in X \) is left (right) generic in \( U \). \( a \) is group-generic if it is both left and right generic.

(ii) A type \( p \) is generic in \( U \) if for every formula \( \varphi \in p \), \( \varphi \) defines a generic subset in \( U \).

Therefore, \( a \in U \) is group-generic of \( U \) over \( A \) if and only if for every \( \varphi \in \text{tp} (a/A) \) such that \( \varphi (\mathcal{M}) \subseteq U \), \( \varphi (\mathcal{M}) \) is generic in \( U \).

**Remark 3.3.2.** Let \( G \) be a group definable over \( A \subseteq M \), and \( a \in G \). If \( a \) is group-generic of \( G \) over \( A \), then \( a \) is generic of \( G \) over \( A \).

Proof. Suppose that there is an \( A \)-definable set \( Y \) with \( a \in Y \) and \( \dim (Y) < \dim (G) \), then \( \dim (Y \cap G) < \dim (G) \), so \( Y \cap G \) cannot be generic in \( G \), but this contradicts the group-genericity of \( a \). □

Note that if \( (R, < ,+ , \cdot) \) is an ordered field, then \( (R, +) \) has no group-generic point over \( \emptyset \). This is because if \( \beta \in R \), then either \( \beta \in (n, \infty) \) or \( \beta \in (-\infty, n) \) for some \( n \in \mathbb{N} \), but none of these intervals is generic in \( (R, +) \). In the case of \( (R^{>0}, \cdot) \), if \( \beta \in (0, n) \) for some \( n \in \mathbb{N} \), then \( \beta \) is not group-generic of \( (R^{>0}, \cdot) \) over \( \emptyset \).
3.3.1 Basics on group-generic points

From now until the end of this chapter, $\mathcal{M}$ is a sufficiently saturated o-minimal expansion of a real closed field.

Below we will discuss some properties of group-generic points, their relationships with generic points as well as some properties of generic points that do not necessarily hold for group-generic points. Saturation and some results in [30] guarantee the existence of group-generic points in definably compact groups (see Prop. 3.3.4).

The following fact is a consequence of [30, Thm. 3.7].

**Fact 3.3.3.** [30] Let $G$ be a definably compact definably connected group definable over $A \subseteq \mathcal{M}$. Then:

(i) The union of two nongeneric definable subsets in $G$ is also nongeneric in $G$.

(ii) The set

$$I = \{X \subseteq G : X \text{ is definable and nongeneric in } G\}$$

is an ideal of $(\text{Def}(G), \cup, \cap)$, the Boolean algebra of definable subsets of $G$.

(iii) There is a complete generic type $p(x) \in S^\mathcal{M}(A)$ in $G$.

**Proposition 3.3.4.** Let $G$ be a definably compact definably connected group definable over $A \subseteq \mathcal{M}$. Then:

(i) If $|A| < \kappa$, then there is a group-generic point $a$ of $G$ over $A$.

(ii) Let $p \in S^\mathcal{M}(A)$ be a generic type in $G$. Then for any $B$ such that $A \subseteq B \subseteq \mathcal{M}$ there is a generic type $q \in S^\mathcal{M}(B)$ in $G$ such that $q \supseteq p$ and $q\vert_A = p$.

**Proof.** (i) By Fact 3.3.3(iii), there is a complete generic type $p(x) \in S^\mathcal{M}(A)$ in $G$. Then, by saturation, there is $a \models p$, so $\text{tp}(a/A) = p$ is generic in $G$; i.e., $a$ is group-generic of $G$ over $A$.

(ii) Suppose that $G = \varphi(\mathcal{M}, a)$ for some $\mathcal{L}_A$-formula $\varphi$ with $a \subseteq A$. Let

$$I_B = \{X \subseteq G : X \text{ is } \mathcal{M}\text{-definable over } B \text{ and nongeneric in } G\}$$

33
and
\[ \Phi_B = \{ \neg \psi \land \varphi : \psi \text{ is a } \mathcal{L}_B\text{-formula that defines in } \mathcal{M} \text{ a set in } \mathcal{I}_B \}. \]

Let \( \overline{p} := p \cup \Phi_B \). Then \( \overline{p} \) is a partial type since if there are \( \{ \theta_i \}_{i<k_1} \subseteq p \), \( \{ \neg \psi_j \land \varphi \}_{j<k_2} \subseteq \Phi_B \) such that \( \bigwedge_{i<k_1} \theta_i \land \bigwedge_{j<k_2} \neg \psi_j \land \varphi \) is not satisfiable, then \( \bigwedge_{i<k_1} \theta_i \rightarrow \bigvee_{j<k_2} \psi_j \) is satisfiable, but \( \bigwedge_{i<k_1} \theta_i \in p \) and by Fact 3.3.3(i), \( \bigvee_{j<k_2} \psi_j \) is nongeneric, so there is a formula in \( p \) that implies a nongeneric formula, which contradicts the genericity of \( p \). Hence, \( \overline{p} \) is finitely satisfiable. Now, let \( q(x) \) be any complete type in \( S^\mathcal{M}(B) \) such that \( q \supseteq \overline{p} \), then \( q \) is a generic type in \( G \) and \( q|_A = p \). This finishes the proof of (ii). \( \square \)

**Corollary 3.3.5.** Let \( G \) be a definably compact definably connected group definable over \( A \subseteq M \) with \( |A| < \kappa \). Let \( a \in G \) be a group-generic element of \( G \) over \( A \), and \( c \) a finite tuple from \( M \). Then,

(i) there is \( a' \in G \) such that \( tp^\mathcal{M}(a'/A) = tp^\mathcal{M}(a/A) \) and \( a' \) is a group-generic element of \( G \) over \( Ac \).

(ii) There is \( c' \) a finite tuple from \( M \) such that \( tp^\mathcal{M}(c'/A) = tp^\mathcal{M}(c/A) \) and \( a \) is a group-generic element of \( G \) over \( Ac' \).

(iii) There is \( c' \) a generic element of \( G \) over \( Aa \) such that \( a \) is group-generic of \( G \) over \( Ac' \).

(iv) Let \( b \in G \) be a group-generic of \( G \) over \( Ac \). If \( c' \) is a finite tuple from \( M \) such that \( tp^\mathcal{M}(c'/Ab) = tp^\mathcal{M}(c/Ab) \), then \( b \) is a group-generic of \( G \) over \( Ac' \).

(v) Let \( b \in G \) be a group-generic of \( G \) over \( A \) and \( a \) a group-generic of \( G \) over \( Ab \). Then there is \( c' \) a finite tuple from \( M \) such that \( tp^\mathcal{M}(c'/A) = tp^\mathcal{M}(c/A) \), and \( b \) and \( a \) are group-generic of \( G \) over \( Ac' \) and \( Abc' \), respectively.

(vi) Let \( b \in G \) be a group-generic of \( G \) over \( A \) and \( a \) a group-generic of \( G \) over \( Ab \). Then there is \( c' \) a generic of \( G \) over \( Aab \) such that \( b \) and \( a \) are group-generic of \( G \) over \( Ac' \) and \( Abc' \), respectively.

**Proof.** (i) Since \( a \) is group-generic of \( G \) over \( A \), \( tp^\mathcal{M}(a/A) \) is a complete generic type in \( G \). Let \( B = Ac \). Then by Prop. 3.3.4(ii), there is a generic type \( q \in S^\mathcal{M}(B) \) in \( G \) such that \( q \supseteq tp^\mathcal{M}(a/A) \) and \( q|_A = tp^\mathcal{M}(a/A) \). As \( \mathcal{M} \) is sufficiently saturated, there is \( a' \subseteq M \) such that \( a' \models q \), thus \( tp^\mathcal{M}(a'/B) = q \). Since \( q|_A = tp^\mathcal{M}(a/A) \), \( tp^\mathcal{M}(a'/A) = tp^\mathcal{M}(a/A) \).
Claim 3.3.6. Let $b \in G$ group-generic of $G$ over $Ac$, and $f \in \text{Aut}(\mathcal{M}/A)$. Then $f(b)$ is group-generic of $G$ over $Af(c)$.

Proof. Let $\varphi(x, a'', f(c))$ be a $L_{\mathcal{M}(\mathcal{M})}$-formula with $a''$ a finite tuple from $A$ such that $\mathcal{M} \models \varphi(f(b), a'', f(c))$ and $\varphi(\mathcal{M}, a'', f(c)) \subseteq G$. We will see that $\varphi(G, a'', f(c))$ is generic in $G$. As $f(b) \models \varphi(x, a'', f(c))$ if and only if $b \models \varphi(x, a'', c)$, then $\varphi(G, a'', c)$ is generic in $G$. From this it is easy to see that $\varphi(G, a'', f(c))$ is also generic in $G$. Thus $f(b)$ is group-generic of $G$ over $Af(c)$. This concludes the proof of Claim 3.3.6. \hfill \square

Now, by (i), there is $a' \subseteq M$ such that $tp^\mathcal{M}(a'/A) = tp^\mathcal{M}(a/A)$ and $a'$ is a group-generic element of $G$ over $Ac$. As $\mathcal{M}$ is sufficiently saturated, $tp^\mathcal{M}(a'/A) = tp^\mathcal{M}(a/A)$ if and only if there is $f \in \text{Aut}(\mathcal{M}/A)$ such that $f(a') = a$. Then, by Claim 3.3.6, $a$ is a group-generic element of $G$ over $Af(c)$, so with $c' = f(c)$ we obtain the desired conclusion.

(iii) Let $c$ be a generic element of $G$ over $A$. By (ii), there is $c' \subseteq M$ such that $tp^\mathcal{M}(c'/A) = tp^\mathcal{M}(c/A)$ and $a$ is a group-generic element of $G$ over $Ac'$. As $a$ is group-generic of $G$ over $Ac'$, $a \downarrow_A c'$. Since $tp^\mathcal{M}(c'/A) = tp^\mathcal{M}(c/A)$ and $c$ is generic of $G$ over $A$, then $c'$ is generic of $G$ over $A$, but $c' \downarrow_A a$, then $c'$ is generic of $G$ over $Aa$.

(iv) As $tp^\mathcal{M}(c'/Ab) = tp^\mathcal{M}(c/Ab)$, then there is $f \in \text{Aut}(\mathcal{M}/Ab)$ such that $f(c) = c'$. Since $b \in G$ group-generic of $G$ over $Ac$, Claim 3.3.6 yields $b = f(b)$ is group-generic of $G$ over $Ac'$.

(v) By (ii), there is $c_1$ a tuple from $M$ such that $tp^\mathcal{M}(c_1/A) = tp^\mathcal{M}(c/A)$ and $b$ is group-generic of $G$ over $Ac_1$. Again by (ii), there is $c'$ a tuple from $M$ such that $tp^\mathcal{M}(c'/Ab) = tp^\mathcal{M}(c_1/Ab)$ and $a$ is group-generic of $G$ over $Abc'$. And by (iv), $b$ is group-generic of $G$ over $Ac'$.

(vi) Let $c$ be a generic element of $G$ over $A$. By (v), there is $c'$ a tuple from $M$ such that $tp^\mathcal{M}(c'/A) = tp^\mathcal{M}(c/A)$, and $b$ and $a$ are group-generic of $G$ over $Ac'$ and $Abc'$, respectively. Since $tp^\mathcal{M}(c'/A) = tp^\mathcal{M}(c/A)$, $a \downarrow_{Ab} c'$, and $b \downarrow_A c'$, then $c'$ is generic of $G$ over $Aab$. \hfill \square
Remark 3.3.7. Let $G$ be a group definable over $A \subseteq M$. If $a$ is group-generic of $G$ over $A$, then $a$ is group-generic of $G$ over $acl^M(A)$.

Proof. Let $c \in acl^M(A)$, and assume that $a \models \varphi(x, c)$ for some $L_c$-formula $\varphi(x, c)$ that defines a subset of $G$. We will see that $\varphi(G, c)$ is generic in $G$. Since $c \in acl^M(A)$ and $\mathcal{M}$ is an ordered structure, then $acl^M = acl^M$ and there is $c' \in A$ such that $\gamma(M, c') = \{c\}$ for some $L$-formula $\gamma$.

Thus, if $\phi(x, c') = (\exists!z) (\varphi(x, z) \land \gamma(z, c'))$, then $a \models \phi(x, c')$.

Since $a$ is group-generic of $G$ over $A$, $\phi(M, c')$ is generic in $G$. Therefore, there are $g_1, \ldots, g_n \in G$ such that for every $g \in G$ there is $g' \models \phi(M, c')$ such that $g = g_i \cdot g'$ for some $i \in \{1, \ldots, n\}$. But $\gamma(M, c') = \{c\}$, then $g' \models \varphi(x, c)$. Thus $\varphi(G, c)$ is generic in $G$.

\[ \square \]

Remark 3.3.8. If $G$ is a group definable over $A \subseteq M$ and $a \in G$ is generic of $G$ over $A$, then $a$ need not be group-generic of $G$ over $A$. For instance, consider a sufficiently saturated real closed field $\mathcal{R} = (R, <, +, 0, \cdot, 1)$ and its additive group $G = (R, +)$. By saturation, there is $\alpha \in \bigcap_{n \in \mathbb{N}} (0, \frac{1}{n})$ (i.e., $\alpha$ is a positive infinitesimal). Since $\dim (\alpha/\emptyset) = tr.deg (\mathbb{Q}(\alpha) : \mathbb{Q}) = 1 = \dim (G)$, then $\alpha$ is generic of $G$ over $\emptyset$, but $\alpha$ is not group-generic of $G$ over $\emptyset$ since $\alpha \in (0, 1)$ and $(0, 1)$ cannot cover $G$ by finitely many group translates.

A similar situation can also be found in a definably compact group. For example, in the same sufficiently saturated real closed field $\mathcal{R} = (R, <, +, 0, \cdot, 1)$ consider a positive infinite element $\beta$; namely, $\beta \in R$ and $\beta > n$ for every $n \in \mathbb{N}$, which exists by saturation. Let $G = ([0, \beta) \subseteq (R, +_{mod})$ with addition modulo $\beta$, which is definably compact. Let $\alpha$ be a positive infinitesimal in $R$, then $\alpha \in G$ and, as above, $\alpha$ is generic of $G$ over $\emptyset$, but $\alpha$ is not group-generic of $G$ over $\emptyset$ since $\alpha \in (0, 1)$ and $(0, 1)$ cannot cover $G$ by finitely many group translates.

Remark 3.3.9. Let $X \subseteq M^n$ definable over $A \subseteq M$. If $b \in X$ is generic of $X$ over $A$ and $a \in X$ is generic of $X$ over $Ab$, then $b$ is generic of $X$ over $Aa$.

Proof. Since $a$ is generic of $X$ over $Ab$, $a \perp_A b$. By the symmetry of the independence (Fact 3.1.3(iii)), $b \perp_A a$, so $b$ is generic of $X$ over $Aa$.

\[ \square \]
Remark 3.3.10. If $G$ is a group definable over $A \subseteq M$, $b \in G$ is group-generic of $G$ over $A$ and $a \in G$ is group-generic of $G$ over $Ab$, then $b$ need not be group-generic of $G$ over $Aa$. For instance, consider a sufficiently saturated real closed field $\mathcal{R} = (R, <, +, 0, \cdot, 1)$. Let $[0, 1) \subseteq R$ and $G = ([0, 1), +_{\text{mod}1})$.

We will find $a, b \in G$ such that $b$ and $a$ are group-generic of $G$ over $A$ and $Ab$, respectively, but with $b$ not group-generic of $G$ over $Aa$.

Let $\varphi$ be a $\mathcal{L}_A$-formula that defines $G$.

Let

$$I_A = \{ \psi : \psi \text{ is a } \mathcal{L}_A\text{-formula and } \psi(R) \subseteq G \text{ is nongeneric in } G \}.$$

Let $\theta_n(x) = 0 < x < \frac{1}{n}$ for $n \in \mathbb{N} \setminus \{0\}$, and let

$$\Gamma_A = \{ \neg \psi \land \varphi, \theta_n : \psi \in I_A, n \in \mathbb{N} \setminus \{0\} \}.$$

$\Gamma_A$ is a partial type because if

$$\left( \bigwedge_{i < k_1} \neg \psi_i \land \varphi \land \bigwedge_{i < k_2} \theta_j \right)(\mathcal{R}) = \emptyset,$$

then $(\bigwedge_{i < k_2} \theta_j)(\mathcal{R}) \subseteq (\bigvee_{i < k_1} \psi_i)(\mathcal{R})$, but the finite union of nongeneric definable subsets in $G$ is nongeneric, so $(\bigvee_{i < k_1} \psi_i)(\mathcal{R})$ cannot contain the generic set $(\bigwedge_{i < k_2} \theta_j)(\mathcal{R})$, thus $\Gamma_A$ is a partial type. Moreover, $\Gamma_A$ is generic in $G$ because every formula in $\Gamma_A$ defines a generic subset in $G$.

Now, let $p \in S^R(A)$ with $p \supseteq \Gamma_A$, then $p$ is also generic in $G$, otherwise if there is $\phi \in p$ defining a nongeneric subset in $G$, then $\neg \phi \land \varphi, \phi \in p$, but this is a contradiction.

By saturation, there is $b \models p$, thus $b$ is group-generic of $G$ over $A$.

In what follows, we will show the existence of a group-generic point $a$ of $G$ over $Ab$ that is also a positive infinitesimal, but $b$ is not group-generic of $G$ over $Aa$. 

37
Let

\[ I_{Ab} = \{ \psi : \text{\( \psi \) is a \( L_{Ab} \)-formula and \( \psi (R) \subseteq G \) is nongeneric in \( G \)\} \} . \]

Let

\[ \Gamma_{Ab} = \{ \neg \psi \land \varphi, \theta_n : \psi \in I_{Ab}, n \in \mathbb{N} \setminus \{0\} \} \]

with \( \theta_n (x) = 0 < x < \frac{1}{n} \) for \( n \in \mathbb{N} \setminus \{0\} \) as above.

As before, \( \Gamma_{Ab} \) is a generic partial type in \( G \) and if \( q \in S^R (Ab) \) with \( q \supseteq \Gamma_{Ab} \), then \( q \) is generic in \( G \).

By saturation, there is \( a \models q \), thus \( a \) is group-generic of \( G \) over \( Ab \).

Notice that \( 0 < b < a \), otherwise if \( a < b \), then \( a \in (0, b) \) and since \( (0, b) \) is an \( Ab \)-definable subset of \( G \), the group-genericity of \( a \) implies that \( (0, b) \) is generic in \( G \), but this is not possible since \( b \) is infinitesimal. Therefore, \( b \in (0, a) \), which is an \( Aa \)-definable interval of infinitesimal length, then \( (0, a) \) cannot be generic in \( G \), so \( b \) is not group-generic in \( G \) over \( Aa \). Thus we finish Remark 3.3.10.

**Fact 3.3.11.** [35, Lemma 2.1] Let \( G \) be a group definable over \( A \subseteq M \). Let \( a, b \in G \), then:

(i) If \( a \) is generic of \( G \) over \( Ab \), then \( ab \) is generic of \( G \) over \( Ab \).

(ii) There are \( b_1, b_2 \in G \) generic of \( G \) over \( Ab \) such that \( b = b_1 b_2 \).

**Claim 3.3.12.** Let \( G \) be a group definable over \( A \subseteq M \). Let \( a, b \in G \), then:

(i) If \( a \) is group-generic of \( G \) over \( Ab \), then \( ab \) is group-generic of \( G \) over \( Ab \).

(ii) If there is a group-generic point of \( G \) over \( Ab \), then there are \( b_1, b_2 \in G \) group-generic of \( G \) over \( Ab \) such that \( b = b_1 b_2 \).

**Proof.** (i) We will show that \( ab \) is group-generic of \( G \) over \( Ab \), so let \( X \subseteq G \) \( Ab \)-definable with \( ab \in X \). Since \( a \in X b^{-1} \) and \( a \) is group-generic of \( G \) over \( Ab \), then \( X b^{-1} \) is generic in \( G \), so is \( X \).
By hypothesis, there is a group-generic point \( b_1 \) of \( G \) over \( Ab \). Since \( b^{-1}_1 \) is also group-generic of \( G \) over \( Ab \), (i) implies \( b_2 = b^{-1}_1 b \) is group-generic of \( G \) over \( Ab \), and \( b = b_1 (b^{-1}_1 b) \).

\[ \square \]

### 3.4 Generic sets in the product group

In this section we prove some properties of generic definable subsets of the product group \( G \times G \) for a definably compact group \( G \). Lemmas 3.4.3 and 3.4.9 will be used in the proof of Theorem 4.3.1.

We first recall the notion of a Keisler measure on \( G \) (which exists by [23, Thm. 7.7]), and a Fubini (or symmetry) Theorem ([23, Prop. 7.5]).

**Definition 3.4.1.** Let \( X \subseteq M^n \) definable.

(i) A Keisler measure \( \mu \) on \( X \) is a finitely additive probability measure on \( Def(X) \) (the set of all definable subsets of \( X \)); i.e., a map \( \mu : Def(X) \to [0,1] \subseteq M \) such that \( \mu(\emptyset) = 0 \), \( \mu(X) = 1 \), and for \( X_1, X_2 \in Def(X) \), \( \mu(X_1 \cup X_2) = \mu(X_1) + \mu(X_2) - \mu(X_1 \cap X_2) \).

(ii) If \( \mu \) is a Keisler measure on a definable group \( G \), \( \mu \) is left (right) invariant if \( \mu(gY) = \mu(Y) \) \( (\mu(Yg) = \mu(Y)) \) for every \( g \in G \) and \( Y \in Def(G) \). \( \mu \) is called generic if: \( \mu(Y) > 0 \) if and only if \( Y \in Def(G) \) and \( Y \) is generic in \( G \).

The following fact gathers Proposition 7.5 and Theorem 7.7 in [23] for the case of a definably compact group.

**Fact 3.4.2.** [23] Let \( G \) be a definably connected definably compact group definable in \( \mathcal{M} \). Then

(i) \( G \) has a unique left invariant Keisler measure, which is also the unique right invariant Keisler measure on \( G \). This measure is also generic.

(ii) If \( \mu \) and \( \lambda \) are two left invariant Keisler measures on \( G \), then

\[ \mu \otimes \lambda = \lambda \otimes \mu \]
and is also left invariant, where for a pair of Keisler measures $\mu_1, \mu_2$ on $G$ we define the Keisler measure $\mu_1 \otimes \mu_2$ on $G \times G$ as follows: for a definable set $D \subseteq G \times G$, $\mu_1 \otimes \mu_2 (D) = \int \mu_1 (D_y) d\mu_2$ where $D_y = \{x \in G : (x, y) \in D\}$. Note that function $y \mapsto \mu_1 (D_y)$ is integrable with respect to $\mu_2$ ([23, 24]).

Lemma 3.4.3. Let $G$ be a definably connected definably compact group definable in $\mathcal{M}$. Let $Z \subseteq G \times G$ be a definable set. For each $y \in G$, let $Z_y = \{x \in G : (x, y) \in Z\}$, $Z_{gen} = \{y \in G : Z_y$ is generic in $G\}$, and $Z_n = \{y \in G : Z_y$ is n-generic in $G\}$. Then $Z$ is generic in $G \times G$ if and only if $Z_{gen}$ is generic in $G$ if and only if there is $n \in \mathbb{N}$ such that $Z_n$ is generic in $G$.

Proof. First, observe that for every $i \in \mathbb{N}$ $Z_i \subseteq Z_{i+1}$, $Z_i$ is definable, and $Z_{gen} = \bigcup_{n \in \mathbb{N}} Z_n$. Let $Z'_1 = Z_1$, and $Z'_n = Z_n \setminus Z_{n-1}$ for $n \geq 2$. Then $Z_{gen} = \bigcup_{n \in \mathbb{N}} Z'_n$. By saturation, $Z_{gen}$ is generic in $G$ if only if there is $n \in \mathbb{N}$ such that $Z_n$ is generic in $G$ if only if there is $n \in \mathbb{N}$ such that $Z'_n$ is generic in $G$.

Second, by Theorem 7.7 in [23], $G$ has a unique generic left invariant Keisler measure $\mu$. Since the Keisler measure $\mu$ is left invariant, so is the Keisler measure $\mu \otimes \mu$ on $G \times G$. Thus again by [23, Theorem 7.7], $\mu \otimes \mu$ is the unique left invariant Keisler measure on $G \times G$.

For a set $X$ we denote by $1_X$ the indicator function of $X$; namely, $1_X (a) = 1$ if $a \in X$ and $1_X (a) = 0$ if $a \notin X$.

$$
\begin{align*}
\mu \otimes \mu (Z) &= \int_G \mu (Z_y) d\mu (y) \\
&= \sum_{n \in \mathbb{N}} \int_{Z'_n} \mu (Z_y) d\mu (y) \\
&= \sum_{n \in \mathbb{N}} \left( \int_{Z'_n} \int_G 1_{Z_y} (x) d\mu (x) \right) d\mu (y) \\
&= \sum_{n \in \mathbb{N}} \int_{Z'_n} \left( \int_G 1_Z (x, y) d\mu (x) \right) d\mu (y).
\end{align*}
$$

(3.4.0.1)

By Proposition 7.5 in [23],

$$
\int_{Z'_n} \left( \int_G 1_Z (x, y) d\mu (x) \right) d\mu (y) = \int_G \left( \int_{Z'_n} 1_Z (x, y) d\mu (y) \right) d\mu (x).
$$
\[ \mu \otimes \mu (Z) = \sum_{n \in \mathbb{N}} \int_G \left( \int_{Z_n} 1_Z(x, y) \, d\mu(y) \right) \, d\mu(x) \]
\[ = \sum_{n \in \mathbb{N}} \int_G \mu(Z^{x,n}) \, d\mu(x), \quad (3.4.0.2) \]

where \( Z^{x,n} = \{ y \in Z_n' : (x, y) \in Z \} \).

Now, we will see that \( \mu \otimes \mu (Z) > 0 \iff (\exists n \in \mathbb{N}) (\mu(Z_n') > 0) \). By Eq. (3.4.0.2), \( \mu \otimes \mu (Z) > 0 \iff (\exists n \in \mathbb{N}) (\exists x \in G) (\mu(Z^{x,n}) > 0) \). Since \( Z^{x,n} \subseteq Z_n' \), then \( \mu \otimes \mu (Z) > 0 \) implies that \( (\exists n \in \mathbb{N}) (\mu(Z_n') > 0) \). For the other direction, let us suppose that \( \mu(Z_n') > 0 \) for some \( n \in \mathbb{N} \). Therefore, \( \int_{Z_n'} \mu(Z_y) \, d\mu(y) > 0 \), so, by Eq. (3.4.0.1), \( \mu \otimes \mu (Z) > 0 \). Thus, \( Z \) is generic in \( G \times G \) if and only if there is \( n \in \mathbb{N} \) such that \( Z_n' \) is generic in \( G \), which is equivalent to \( Z_{gen} \) being generic in \( G \) by the first part of this proof.

Observe that an analogous result can be proved in the same way for fibers of elements in the first component of \( G \times G \).

**Corollary 3.4.4.** Let \( G \) be a definably connected definably compact group definable in \( M \). If \( b \in G \) is group-generic of \( G \) over \( A \) and \( a \in G \) is group-generic of \( G \) over \( Ab \), then \( (a, b) \in G \times G \) is group-generic of \( G \times G \) over \( A \).

**Proof.** Let \( Z \subseteq G \times G \) be \( A \)-definable with \( (a, b) \in Z \). Since \( a \in Z_b = \{ x \in G : (x, b) \in Z \} \), which is \( Ab \)-definable, then \( Z_b \) is generic in \( G \), so \( b \in Z_{gen} = \bigcup_{n \in \mathbb{N}} Z_n \), where \( Z_n = \{ y \in G : Z_y \text{ is } n\text{-generic in } G \} \). Therefore, there is \( n \in \mathbb{N} \) such that \( b \in Z_n \). As \( Z_n \) is \( A \)-definable and \( b \) is group-generic of \( G \) over \( A \), then \( Z_n \) is generic in \( G \). By Lemma 3.4.3, this is equivalent to \( Z \) being generic in \( G \times G \). Then \( (a, b) \in G \times G \) is group-generic of \( G \times G \) over \( A \).

**Remark 3.4.5.** Let \( X \subseteq M^n \) definable over \( A \subseteq M \). If \( b \in X \) is generic of \( X \) over \( A \) and \( a \in X \) is generic of \( X \) over \( Ab \), then \( (a, b) \in X \times X \) is generic of \( X \times X \) over \( A \).

**Proof.** It follows directly from the additivity property of the (acl-)dimension (Fact 3.1.3(ii)).
Now, we will show that for a definably connected definably compact group $G$ any definable generic set in $G \times G$ contains a definable generic box.

Recall that every Hausdorff locally compact group $G$ carries a natural measure called the Haar measure. A left Haar measure $\mathfrak{m}$ on $G$ is a measure on the Borel algebra (namely, the $\sigma$-algebra generated by all open sets of $G$) that is left invariant (i.e., $\mathfrak{m}(gX) = \mathfrak{m}(X)$ for every $g \in G$ and Borel set $X$), finite on every compact subset of $G$, and positive for every non-empty open subset of $G$. By the Theorem of Haar, $G$ has, up to a positive multiplicative constant, a unique nontrivial left Haar measure. If in addition $G$ is compact, then their left Haar measures coincide with their right Haar measures, and since $\mathfrak{m}(G) < \infty$, we can naturally choose a normalized Haar measure on $G$; namely, $\mathfrak{m}(G) = 1$.

**Definition 3.4.6.** Let $G$ be a type-definable group. $G$ is compactly dominated by $(H, \mathfrak{m}, \pi)$, where $H$ is a compact group, $\mathfrak{m}$ is the unique normalized Haar measure on $H$ and $\pi : G \to H$ is a surjective group homomorphism, if for any definable $Y \subseteq G$ and for every $c \in H$ outside a set of $\mathfrak{m}$ measure zero, either $\pi^{-1}(c) \subseteq Y$ or $\pi^{-1}(c) \subseteq G \setminus Y$; namely,

$$\mathfrak{m}\left(\{c \in H : \pi^{-1}(c) \cap Y = \emptyset \text{ and } \pi^{-1}(c) \cap (G \setminus Y) = \emptyset\}\right) = 0.$$

**Fact 3.4.7.** [20, 23] Let $G$ be a definably connected definably compact group definable in $\mathcal{M}$, then $G$ is compactly dominated by $(G/G^{00}, \mathfrak{m}, \pi)$ where $\mathfrak{m}$ is the Haar measure of the compact group $G/G^{00}$ with its logic topology, and $\pi : G \to G/G^{00}$ is the canonical surjective homomorphism.

As a consequence of compact domination, Berarducci proved in [3] the following fact.

**Fact 3.4.8.** [3, Proposition 2.1] Let $G$ be a definably connected definably compact group definable in $\mathcal{M}$. If $X \subseteq G$ is definable and generic in $G$, then some left group translate (equivalently right group translate) of $X$ contains $G^{00}$.

**Lemma 3.4.9.** Let $G$ be a definably connected definably compact group definable in $\mathcal{M}$. Let $Z$ be a definable generic subset in $G \times G$. Then there are definable sets $A, B \subseteq G$ generic in $G$ such that $A \times B \subseteq Z$.

**Proof.** By [3, Proposition 2.1], there is $\overline{g} = (g_1, g_2) \in G \times G$ such that $G^{00} \times G^{00} \subseteq Z \cdot \overline{g}$. Since $G^{00} \times G^{00}$ is a type-definable set and $Z \cdot \overline{g}$ is definable, saturation yields that
there are definable $A^*, B^* \subseteq G$ such that $G^{00} \subseteq A^*, B^*$ and $A^* \times B^* \subseteq Z \cdot g$. Let $A = A^* \cdot g_1^{-1}$ and $B = B^* \cdot g_2^{-1}$, then $A \times B \subseteq Z$ and $A, B$ are both definable and generic in $G$. This ends the proof of the lemma.
Chapter 4

Semialgebraically compact semialgebraic groups and algebraic groups over a real closed field

Assume throughout this chapter that \( \mathcal{R} = (R, <, +, \cdot) \) is a sufficiently saturated real closed field.

The goal of this chapter is to prove Theorem 4.3.1. This theorem states that any definably compact definably connected group \( G \) definable in \( \mathcal{R} \) is locally homomorphic to the set of \( R \)-points of some \( R \)-algebraic group through an injective semialgebraic map whose domain is generic in \( G \).

The proof of this result applies the properties of group-generic points, generic points, and generic sets discussed in Chapter 3 in an adaptation of the proof of Hrushovski and Pillay of Theorem A (see Fact 1.0.1). Note that along the proof of [21, Thm. A] there are several steps where the authors replace a \( k \)-definable set by a smaller one, which might not be defined over \( k \). For our purpose we need to do this more carefully.

Before the proof of Theorem 4.3.1, we will first present a group configuration proposition for definably compact groups (Proposition 4.1.2), and then the definition of local homomorphism and some of its properties.
4.1 A group configuration proposition for definably compact groups

For the proof of the next proposition we will adapt the notion of geometric structure and substructure given in [21, Chapter 2]. Therefore, for the real closed field \( \mathcal{R} = (\mathbb{R}, <, +, \cdot) \), if \( \mathcal{L} = \{+, \cdot\} \), then the algebraic closure \( D = R(\sqrt{-1}) \) of \( R \) is a geometric structure, and \( R \) viewed as an \( \mathcal{L} \)-structure is a geometric substructure of \( D \) and therefore satisfies the following:

(i) the algebraic closures of \( A \subseteq R \) in \( R \) in the model-theoretic and algebraic senses coincide, and for every \( A \subseteq R \) the algebraic closure of \( A \) in \( R \) in the sense of the \( \mathcal{L} \)-structure \( R \) is precisely \( R \cap \text{acl}^D(A) \),

(ii) \( R \) is definably closed in \( D \); that is, if \( b \in D \) and \( b \in \text{dcl}^D(R) \), then \( b \in R \), and

(iii) for each \( \mathcal{L} \)-formula \( \varphi(x, y) \) there is some \( N < \omega \) such that any model \( R_1 \) of \( Th(R) \) and every \( b \in R_1 \), if \( \varphi(x, y) \) defines a finite subset of \( R_1 \), then it defines a set with at most \( N \) elements.

We also adapt the same notation of Hrushovski and Pillay in [21], and we recall it below.

**Notation 4.1.1.** Let \( A \subseteq R \) and \( a \) a finite tuple from \( R \). \( \text{tp}(a/A) \) denotes \( \text{tp}^R(a/A) \), \( \text{dcl}(A) \) denotes \( \text{dcl}^R(A) \).

\( \text{qftp}(a/A) \) denotes \( \text{qftp}^D(a/A) \) that is the set of quantifier-free \( \mathcal{L}_A \)-formulas satisfied by \( a \) in \( D \). \( \text{qfdcl}(A) \) denotes \( \text{qfdcl}^D(A) \) that is the set of elements of \( D \) definable over \( A \) by quantifier-free formulas, but since \( R \) is definably closed in \( D \), so \( \text{qfdcl}^D(A) \subseteq R \). Note that since \( D \) has quantifier elimination, \( \text{qfdcl}^D(A) = \text{dcl}^D(A) \). Finally, by \( \text{acl}(A) \) we denote \( \text{acl}^D(A) \).

Recall that a group \( H \) definable in an algebraically closed field \( D \) is \( D \)-definably connected if there is no proper nontrivial \( D \)-definable subgroup of \( H \) of finite index.

**Proposition 4.1.2.** Let \( D = R(\sqrt{-1}) \) be the algebraic closure of the sufficiently saturated real closed field \( \mathcal{R} = (\mathbb{R}, <, +, \cdot) \). Let \( G \) be a definably compact definably connected group definable in \( \mathcal{R} \). Then there are a finite subset \( A \subseteq R \) over which \( G \) is defined, a definably connected group \( H \) definable in \( D \) over \( A \), points \( a, b, c \) of \( G \) and points \( a', b', c' \) of \( H(R) \) such that

45
(i) \( a \cdot b = c \) (in \( G \)) and \( a' \cdot b' = c' \) (in \( H \)),

(ii) \( acl(aA) = acl(a'A), \ acl(bA) = acl(b'A) \) and \( acl(cA) = acl(c'A) \),

(iii) \( b \) is a group-generic point of \( G \) over \( A \), \( a \) is a group-generic point of \( G \) over \( Ab \),

(iv) \( a' \) and \( b' \) are generic points of \( H(R) \) over \( A \) and are independent with each other over \( A \).

Note that \( a' \) and \( b' \) are generic, but need not be group-generic of \( H(R) \) over \( A \).

Proof. This proof is essentially the same as that of [21, Proposition 3.1] (see Proposition A.0.9 in Appendix A), what is new is that we have to prove that the points \( a \) and \( b \) introduced below remain group-generic of \( G \) over each of the sets of parameters defined by Hrushovski and Pillay in their proof. To achieve this we summarize without proof the unmodified parts of the proof of [21, Proposition 3.1] and just focus on the new parts. A reader interested in the parts of the proof of [21, Proposition 3.1] which we skip can find them in Appendix A. We refer the reader to [21] for appropriate model-theoretic background.

The first part of the proof of Proposition 3.1 in [21] is devoted to yielding a set-up in which [21, Proposition 1.8.1] (see Fact A.0.8 in Appendix A) can be applied, and get with this the existence of the connected group \( H \) definable in \( D \) mentioned in the conclusion of Proposition 3.1. This is done through a series of lemmas and observations.

Let us start with a finite subset \( A_0 \) of \( R \) over which \( G \) and its group operation are defined. Let \( \dim(G) = n \). By Proposition 3.3.4(i), there is \( b \in G \) group-generic of \( G \) over \( A_0 \). By Prop. 3.3.4(ii) and saturation, there is \( a \in G \) group-generic of \( G \) over \( A_0 \cup \{b\} \), then \( a \searrow_{A_0} b \). Let \( c = a \cdot b \), then, by Claim 3.3.12, \( c \) is group-generic of \( G \) over \( A_0 \cup \{b\} \). And also \( \dim(a, b, c/A_0) = \dim(a, b/A_0) = 2n \).

In \( R \), \( c \in dcl(a, b, A_0) \) and \( b \in dcl(a, c, A_0) \). Thus we start with three group-generic points of \( G \) such that each pair of them are independent (over some set of parameters) and define the third in \( R \). As Hrushovski and Pillay point out in [21], the key is to modify those points by points in \( R \) such that two of them define the third in the structure \( D \); namely, that \( dcl \) is replaced by \( qfdcl \) in order to lay the foundations to apply [21, Prop. 1.8.1].
Lemma 4.1.3. There are a finite subset $A_2$ of $R$, containing $A_0$, and tuples $a, b_1, c_1$ in $R$ such that

(i) $b$ and $a$ are group-generic of $G$ over $A_2$ and $A_2b$, respectively,
(ii) $acl(a, A_2) = acl(a_1, A_2)$, $acl(b, A_2) = acl(b_1, A_2)$, $acl(c, A_2) = acl(c_1, A_2)$,
(iii) $b_1 \in qfdcl(a_1, c_1, A_2)$, and $c_1 \in qfdcl(a_1, b_1, A_2)$.

Proof. The only thing we need to prove here is the existence of elements $x', z_1$ from $R$ satisfying the same conditions of the $x'$ and $z_1$ of Hrushovski and Pillay in their original proof of [21, Lemma 3.2] (see Lemma A.0.10 in Appendix A) such that $b$ and $a$ are group-generic of $G$ over $A_2 = A_0 x' z_1$ and $A_2 b$, respectively, because the rest of the proof is exactly the same as that of [21, Lemma 3.2].

Claim 4.1.4. (i) There is a generic point $x'$ of $G$ over $A_0 b$ such that if $A_1 = A_0 x'$, then $b$ is group-generic of $G$ over $A_1$ and $a$ is group-generic of $G$ over $A_1 b$.

(ii) Consider $A_1$ as in (i), then there is a generic point $z_1$ of $G$ over $A_1 b$ such that if $A_2 = A_1 z_1$, then $b$ is group-generic of $G$ over $A_2$ and $a$ is group-generic of $G$ over $A_2 b$.

Proof. (i) Since $b$ is group-generic of $G$ over $A_0$ and $a$ is group-generic of $G$ over $A_0 b$, then Corollary 3.3.5(vi) yields the existence of a generic element $x'$ of $G$ over $A_0 x'$ such that $b$ is group-generic of $G$ over $A_0 x'$ and $a$ is group-generic of $G$ over $A_0 x' b$.

(ii) From the conclusion of (i) and Corollary 3.3.5(vi), we get the existence of a generic point $z_1$ of $G$ over $A_1 b$ such that $b$ is group-generic of $G$ over $A_1 z_1$ and $a$ is group-generic of $G$ over $A_1 z_1 b$.

This ends the proof of Lemma 4.1.3.

Now, let $a_1, b_1, c_1$ and $A_2$ be as given by Lemma 4.1.3, and $A = acl(A_2) \cap R$. Therefore, $a_1, b_1, c_1$ each have dimension $n$ over $A$. Since $acl(A_2) \cap R = acl^R(A_2)$, then Remark 3.3.7 implies that $b$ and $a$ are also group-generic of $G$ over $A$ and $Ab$, respectively.
Remark A.0.15 yields that \( \text{qftp} \left( b_1, c_1/A, a_1 \right) \) is stationary, and hence we can define the canonical base \( \sigma \) of \( \text{qftp} \left( b_1, c_1/A, a_1 \right) \). Then \( \sigma \in \text{qfdcl} \left( A, a_1 \right) \). Since \( R \) is definably closed in \( D, \sigma \in R \).

Let

\[
\tau = \text{qftp} \left( \sigma/A \right), \quad q_1 = \text{qftp} \left( b_1/A \right), \quad \text{and} \quad q_2 = \text{qftp} \left( c_1/A \right),
\]

then \( \dim \left( q_1 \right) = \dim \left( q_2 \right) = n \).

By Remark 3.4 in [21], \( r \) is stationary, \( \dim \left( r \right) = n, \sigma \downarrow_A b_1, \sigma \downarrow_A c_1, b_1 \in \text{qfdcl} \left( \sigma, c_1, A \right) \), and \( c_1 \in \text{qfdcl} \left( \sigma, b_1, A \right) \). Therefore, there is some \( A \)-definable partial function in the sense of \( D \), say \( \mu \), such that \( c_1 = \mu \left( \sigma, b_1 \right) \). And note that whenever \( \sigma' \models r \) and \( b_1' \models q_1 \) with \( \sigma' \downarrow_A b_1' \), then \( \mu \left( \sigma', b_1' \right) \) is well-defined, realises \( q_2 \) and is independent with each of \( \sigma', b_1' \) over \( A \). Similarly, as \( b_1 \in \text{qfdcl} \left( \sigma, c_1, A \right) \), there is some \( A \)-definable partial function in the sense of \( D \), say \( \nu \), such that \( b_1 = \nu \left( \sigma, c_1 \right) \). And note that whenever \( \sigma' \models r \) and \( c_1' \models q_2 \) with \( \sigma' \downarrow_A c_1' \), then \( \nu \left( \sigma', c_1' \right) \) is well-defined, realises \( q_1 \) and is independent with each of \( \sigma', c_1' \) over \( A \).

Now, let \( \sigma_1, \sigma_2 \models r \) with \( \sigma \downarrow_A \sigma_1, \sigma_2 \) and \( \sigma_1, \sigma_2 \in R \). Let \( b_2 \models q_1 \) such that \( b_2 \downarrow_A \left\{ \sigma_1, \sigma_2 \right\} \) and \( b_2 \in R \).

Then \( \mu \left( \sigma_1, b_2 \right) \) is defined, realises \( q_2 \), and is independent with \( \sigma_2 \) over \( A \). Therefore, \( \nu \left( \sigma_2, \mu \left( \sigma_1, b_2 \right) \right) \) is defined and realises \( q_1 \). Denote \( \nu \left( \sigma_2, \mu \left( \sigma_1, b_2 \right) \right) \) by \( b_3 \).

By Remark 3.6 in [21], \( b_3 \in \text{qfdcl} \left( \sigma_1, \sigma_2, b_2, A \right) \), \( b_2 \in \text{qfdcl} \left( \sigma_1, \sigma_2, b_3, A \right) \), each of \( b_2, b_3 \) is independent with \( \left\{ \sigma_1, \sigma_2 \right\} \) over \( A \), and \( \text{qftp} \left( b_2, b_3/\sigma_1, \sigma_2, A \right) \) is stationary. Then we can define the canonical base of \( \text{qftp} \left( b_2, b_3/\sigma_1, \sigma_2, A \right) \) and denote it by \( \tau \). Then \( \tau \in \text{qfdcl} \left( \sigma_1, \sigma_2, A \right) \), so \( \tau \in R \).

Let \( s = \text{qftp} \left( \tau/A \right) \). By [21, Lemma 3.7], \( \dim \left( s \right) = n \). As was proved in Remark 3.4 in [21], we have that \( b_3 \in \text{qfdcl} \left( \tau, b_2, A \right) \), \( b_2 \in \text{qfdcl} \left( \tau, b_3, A \right) \); moreover, \( \tau \) is independent with each of \( b_2, b_3 \) over \( A \). Therefore, there is some \( A \)-definable partial function \( \mu' \) in the sense of \( D \) such that \( b_3 = \mu' \left( \tau, b_2 \right) \), and whenever \( \tau' \models s \) and \( b_1' \models q_1 \) with \( \tau' \downarrow_A b_1' \), then \( \mu' \left( \tau', b_1' \right) \) is well-defined and realises \( q_1 \).

At this stage Hrushovski and Pillay obtain two \( n \)-dimensional stationary types \( s \) and \( q_1 \) over \( A \) that satisfy the hypothesis of [21, Proposition 1.8.1]. Moreover, the functions \( f \) and \( g \), which are quantifier-free definable in \( D \) over \( A \), in the hypothesis of Prop. 1.8.1 in [21] correspond to the functions \( f \) in [21, Lemma 3.8] (see Lemma A.0.21 in Appendix A) and \( \mu' \), respectively.

48
Next comes the application of [21, Prop. 1.8.1]. Let $H$, $X$, $h_1$, and $h_2$ as given by Prop. 1.8.1 in [21]. We can assume that $h_1, h_2$ are both the identity function. Thus $H$ is a connected group definable in $D$ over $A$ with generic type $s$, $X$ is a set definable in $D$ over $A$ with generic type $q_1$, and there is a transitive group action $\Lambda : H \times X \to X : (h, x) \mapsto \Lambda (h, x)$, which is also definable in $D$ over $A$.

Note that since $\tau \in H(R)$, $\tau \models s$, and $R$ viewed as an $\{+\cdot\}$-structure is a geometric substructure of $D$, then $\dim (H(R)) = n$. Similarly, from $b_1 \in X(R)$ and $b_1 \models q_1$, we have $\dim (X(R)) = n$.

Moreover, we have the following:

(i) for $\tau_1, \tau_2 \models s$ with $\tau_1 \downarrow_A \tau_2$, the product $\tau_1 \cdot \tau_2$ in the group $H$ is exactly $f (\tau_1, \tau_2)$, and

(ii) for any $\tau \models s$ and $b \models q_1$ with $\tau \downarrow_A b$, $\Lambda (\tau, b)$ is exactly $\mu'(\tau, b)$.

Since $\tau \in \text{qfdcl} (\sigma_1, \sigma_2, A)$, there is some $A$-definable partial function $\xi$ in the sense of $D$ such that $\tau = \xi (\sigma_1, \sigma_2)$. And note that whenever $\sigma'_1, \sigma'_2 \models r$ with $\sigma'_1 \downarrow_A \sigma'_2$, then $\xi (\sigma'_1, \sigma'_2)$ is well-defined, realises $s$ and is independent with each of $\sigma'_1, \sigma'_2$ over $A$.

Finally, in the last part of this proof we introduce some new sets of parameters, define the points $a', b', c'$ generic in $H(R)$, and establish interalgebraic between them and the points $a, b, c$.

Let $\sigma, b_1, c_1$ be as fixed after the proof of Lemma 4.1.3, which are all in $R$.

**Claim 4.1.5.** There is a tuple $\sigma_1$ from $R$ such that $\text{qftp} (\sigma_1/A) = \text{qftp} (\sigma/A)$, $\sigma_1 \downarrow_A \{\sigma, b_1, c_1\}$, and $b$ and $a$ are group-generic of $G$ over $\text{acl} (A\sigma_1) \cap R$ and $(\text{acl}(A\sigma_1) \cap R) \cup \{b\}$, respectively.

**Proof.** Since $r = \text{qftp} (\sigma/A)$ is stationary and every definable set in $R$ has generic points in $R$, then there is $\tilde{\sigma}_1$ a tuple from $R$ such that $\text{qftp} (\tilde{\sigma}_1/A) = \text{qftp} (\sigma/A)$ and $\tilde{\sigma}_1 \downarrow_A \{\sigma, b_1, c_1\}$. Then, by Corollary 3.3.5(v), there is $\sigma_1 \subseteq R$ such that $\text{tp} (\tilde{\sigma}_1/A) = \text{tp} (\sigma_1/A)$, $b$ and $a$ are group-generic of $G$ over $A\sigma_1$ and $A\sigma_1 b$, respectively. We will see that $\sigma_1$ satisfies the same properties of $\tilde{\sigma}_1$. 

49
First, since \( tp(\tilde{\sigma}/A) = tp(\sigma_1/A) \) and \( \text{qftp}(\tilde{\sigma}/A) = \text{qftp}(\sigma/A) \), so \( \text{qftp}(\sigma_1/A) = \text{qftp}(\sigma/A) \). Second, from the construction throughout the proof of [21, Proposition 3.1], we have that \( \{\sigma, b_1, c_1\} \) and \( \{a, b\} \) are interalgebraic over \( A \). Then, \( \sigma_1 \downarrow_A \{\sigma, b_1, c_1\} \) if and only if \( \sigma_1 \downarrow_A \{a, b\} \). And note that since \( b \downarrow_A \sigma_1 \) and \( a \downarrow_{Ab} \sigma_1 \), then \( \sigma_1 \downarrow_A \{a, b\} \).

By Remark 3.3.7, if \( b \) and \( a \) are group-generic of \( G \) over \( A\sigma_1 \) and \( A\sigma_1b \), respectively, then \( b \) and \( a \) are group-generic of \( G \) over \( \text{acl}(A, \sigma_1) \cap R \) and \( (\text{acl}(A, \sigma_1) \cap R) \cup \{b\} \), respectively. This completes the proof of the claim. \( \square \)

Let \( \sigma_1 \) be as given by Claim 4.1.5. Then \( \nu(\sigma_1, c_1) \models q_1 \), is in \( R \), and \( \nu(\sigma_1, c_1) \downarrow_A \sigma_1 \). Let \( c_2 = \nu(\sigma_1, c_1) \). Also, we get \( \xi(\sigma_1, \sigma) \models s \), \( \xi(\sigma_1, \sigma) \downarrow_A \sigma_1 \), and is in \( H(R) \). Let \( \tau = \xi(\sigma_1, \sigma) \), and \( A_1 = \text{acl}(A, \sigma_1) \cap R \). Then so far we have that:

1. \( b \) and \( a \) are group-generic of \( G \) over \( A_1 \) and \( A_1b \), respectively,
2. \( \text{acl}(A_1, a) = \text{acl}(A_1, \tau) \), \( \text{acl}(A_1, c) = \text{acl}(A_1, c_2) \), and
3. \( \tau \models s \), \( b_1 \models q_1 \), and \( c_2 \models q_1 \).

We complete the proof of this proposition below.

Since \( R \) is definably closed in \( D \), for every \( \tau' \in H(R) \) and every \( \beta \in X(R) \), \( \Lambda(\tau', \beta) \in X(R) \). Moreover, \( H(R) \) acts on \( X(R) \) by the group action \( \Lambda \) restricted to \( H(R) \times X(R) \), which is definable in \( R \) over \( A \).

Let us define a relation \( \sim \) on \( X(R) \). For \( \beta_1, \beta_2 \in X(R) \) we say \( \beta_1 \sim \beta_2 \) if and only if \( \beta_1 \) and \( \beta_2 \) are both in the same \( H(R) \)-orbit, namely if \( \beta_1 \in \Lambda(H(R), \beta_2) \). Then \( \sim \) is an equivalence relation on \( X(R) \) definable in \( R \) over \( A \subseteq A_1 \).

Since \( R \) is o-minimal and has elimination of imaginaries, [29, Corollary 4.7] implies that there are at most finitely many \( \sim \)-classes whose dimension equals \( \dim(X(R)) \). Therefore, for every \( \beta \) generic of \( X(R) \) over \( A_1 \), the equivalence class of \( \beta \) under \( \sim \), denoted \( [\beta] \), has dimension \( n \) and is a definable set in \( R \) over \( A \subseteq A_1 \).

Now, recall that \( b_1 \) is generic of \( X(R) \) over \( A_1 \) and \( b_1 \models q_1 \), then \( [b_1] \) is an \( n \)-dimensional set definable in \( R \) over \( A_1 \).
Let $b'_2$ be a generic element of $[b_1]$ over $A_1$. Then by Corollary 3.3.5(v), there is a tuple $b_2$ from $R$ such that $\text{tp}(b_2/A_1) = \text{tp}(b'_2/A_1)$, and $b$ and $a$ are group-generic of $G$ over $A_1b_2$ and $A_1bb_2$, respectively. Since $b'_2 \in [b_1]$ and $[b_1]$ is defined over $A_1$, then $b_2 \in [b_1]$. Thus, there is $\tau_1 \in H(R)$ such that $b_2 = \Lambda (\tau_1^{-1}, b_1)$.

Since $a$ is also generic of $G$ over $A_1bb_2$, then $a \perp_{A_1} \{b, b_2\}$; thus $b_2 \perp_{A_1} a$. Also, as $b$ is also generic of $G$ over $A_1b_2$, $b_2 \perp_{A_1} b$. Therefore, $b_2 \perp_{A_1} \{a, b\}$, thus $b_2$ is generic of $X(R)$ over $A_1ab$.

Now, since $\text{acl}(A_1, a) = \text{acl}(A_1, \tau)$ and $\text{acl}(A_1, b) = \text{acl}(A_1, b_1)$, then $b_2 \perp_{A_1} \{\tau, b_1\}$.

**Claim 4.1.6.** (i) $\tau_1$ is generic of $H(R)$ over $A_1\tau b_1$, and

(ii) $\tau_1$ is generic of $H(R)$ over $A_1\tau b_2$.

(iii) $\tau$ is generic of $H(R)$ over $A_1\tau_1 b_2$.

**Proof.** (i) Note that

$$\dim (\tau_1, b_2/A_1, \tau, b_1) = \dim (b_2/A_1, \tau, b_1) + \dim (\tau_1/A_1, \tau, b_1, b_2) = n + \dim (\tau_1/A_1, \tau, b_1, b_2).$$

Also, $\dim (\tau_1, b_2/A_1, \tau, b_1) = \dim (\tau_1/A_1, \tau, b_1)$ since $b_2 = \Lambda (\tau_1^{-1}, b_1)$. Therefore,

$$\dim (\tau_1/A_1, \tau, b_1) = n.$$

(ii) First, observe that as $b_2 \in \text{acl}(A_1, \tau_1, b_1)$, then

$$\dim (\tau_1, b_2, b_1/A_1, \tau) = \dim (\tau_1, b_1/A_1, \tau) = \dim (\tau_1/A_1, \tau, b_1) + \dim (b_1/A_1, \tau) = 2n.$$

Also,

$$\dim (\tau_1, b_2, b_1/A_1, \tau) = \dim (b_2/A_1, \tau) + \dim (\tau_1/A_1, \tau, b_2) = n + \dim (\tau_1/A_1, \tau, b_2).$$

Hence, $\dim (\tau_1/A_1, \tau, b_2) = n$. 

51
(iii) By (ii), \( \tau_1 \downarrow_{A_1, b_2} \tau \), and since \( b_2 \downarrow_{A_1} \tau \), then \( \tau \downarrow_{A_1} \{b_2, \tau_1\} \). This finishes this proof.

Claim 4.1.7.  
(i) \( acl(A_1, b_2, a) = acl(A_1, b_2, \tau) \),

(ii) \( acl(A_1, b_2, b) = acl(A_1, b_2, \tau_1) \), and

(iii) \( acl(A_1, b_2, c) = acl(A_1, b_2, \tau \cdot \tau_1) \).

Proof. (i) It follows from \( acl(A_1, a) = acl(A_1, \tau) \).

(ii) First, we will see that \( \tau_1 \in acl(A_1, b_2, b_1) \).

\[
\dim (\tau_1, b_2/A_1, b_1) = \dim (b_2/A_1, b_1) + \dim (\tau_1/A_1, b_2, b_1)
\]
\[
= n + \dim (\tau_1/A_1, b_2, b_1).
\]

Also,

\[
\dim (\tau_1, b_2/A_1, b_1) = \dim (\tau_1/A_1, b_1) + \dim (b_2/A_1, b_1, \tau_1)
\]
\[
= n.
\]

Then, \( \tau_1 \in acl(A_1, b_2, b_1) \), and since \( b \) and \( b_1 \) are interalgebraic over \( A_1 \), so \( acl(A_1, b_2, \tau_1) \subseteq acl(A_1, b_2, b) \).

Additionally, \( b_1 = \Lambda (\tau_1, b_2) \), then \( b_1 \in acl(A_1, b_2, \tau_1) \). So \( acl(A_1, b_2, b) \subseteq acl(A_1, b_2, \tau_1) \). This completes (ii).

(iii) First, we will see that \( \Lambda (\tau \cdot \tau_1, b_2) = c_2 \). From the properties of the action and the maps \( \nu, \mu \), and \( \xi \), we have

\[
\Lambda (\tau \cdot \tau_1, b_2) = \Lambda (\tau \cdot \tau_1, \Lambda (\tau_1^{-1}, b_1))
\]
\[
= \Lambda (\tau, \Lambda (\tau_1 \cdot \tau_1^{-1}, b_1))
\]
\[
= \Lambda (\tau, b_1),
\]
\[
c_2 = \nu (\sigma_1, c_1)
\]
\[
= \nu (\sigma_1, \mu (\sigma, b_1))
\]
\[
= \Lambda (\xi (\sigma_1, \sigma), b_1)
\]
\[
= \Lambda (\tau, b_1).
\]
Then, $\Lambda (\tau \cdot \tau_1, b_2) = c_2$.

We will see that $\tau \cdot \tau_1 \in \text{acl} (A_1, b_2, c_2)$. By Fact 3.3.11, since $\tau$ is generic of $H (R)$ over $A_1 \tau_1 b_2$, then $\tau \cdot \tau_1$ is generic of $H (R)$ over $A_1 \tau_1 b_2$.

Now, since $b_2 \downarrow_{A_1} \{a, b\}$ and $\text{acl} (A_1, c) = \text{acl} (A_1, c_2)$, $c_2 \downarrow_{A_1} b_2$, and thus we get

$$\dim (\tau \cdot \tau_1, c_2/A_1, b_2) = \dim (c_2/A_1, b_2) + \dim (\tau \cdot \tau_1/A_1, b_2, c_2)$$

$$= n + \dim (\tau \cdot \tau_1/A_1, b_2, c_2).$$

Also,

$$\dim (\tau \cdot \tau_1, c_2/A_1, b_2) = \dim (\tau \cdot \tau_1/A_1, b_2) + \dim (c_2/A_1, b_2, \tau \cdot \tau_1)$$

$$= n.$$

Thus, $\tau \cdot \tau_1 \in \text{acl} (A_1, b_2, c_2)$, and thus $\text{acl} (A_1, b_2, \tau \cdot \tau_1) \subseteq \text{acl} (A_1, b_2, c)$. Finally, as $c_2 = \Lambda (\tau \cdot \tau_1, b_2)$ and $\text{acl} (A_1, c) = \text{acl} (A_1, c_2)$, then $\text{acl} (A_1, b_2, c) \subseteq \text{acl} (A_1, b_2, \tau \cdot \tau_1)$, this ends this proof.

Let $A_2 = \text{acl} (A_1, b_2) \cap R$, $a' = \tau$, $b' = \tau_1$, and $c' = a' \cdot b'$ the product of $a'$ by $b'$ in $H$. So far we have proved that:

(i) By Remark 3.3.7, $b$ and $a$ are group-generic of $G$ over $A_2$ and $A_2 b$, respectively,

(ii) $a'$ and $b'$ are generic of $H (R)$ over $A_2$ and $a' \downarrow_{A_2} b'$,

(iii) $\text{acl} (A_2, a) = \text{acl} (A_2, a')$, $\text{acl} (A_2, b) = \text{acl} (A_2, b')$, and $\text{acl} (A_2, c) = \text{acl} (A_2, c')$.

Finally, let $A$ any finite subset of $A_2$ over which $G$ and $H$ are defined with the obtained properties. This concludes the proof of Proposition 4.1.2.

4.2 Local homomorphisms

**Definition 4.2.1.** Let $G_1$ and $G_2$ be two topological groups, $X \subseteq G_1$ a neighborhood of the identity of $G_1$, and $\theta : X \to G_2$ a map. $\theta$ is called a local homomorphism if
Let \( \theta : X \subseteq G_1 \to G_2 \) be an injective local homomorphism between the groups \( G_1, G_2 \). Then \( \theta^{-1} : \theta(X) \to X \) need not be a local homomorphism; for instance, consider the groups \( G_1 = (\mathbb{R}, +), G_2 = ([0, 1], +_{\text{mod} 1}) \), and \( \theta : [-\frac{1}{8}, \frac{1}{8}] \subseteq \mathbb{R} \to [0, 1) : t \mapsto t \text{ mod } 1 \). Then \( \theta \) is an injective local homomorphism, but \( \theta^{-1} : [0, \frac{3}{8}] \cup [\frac{3}{8}, 1) = \theta \left( \left[ -\frac{1}{8}, \frac{1}{8} \right] \right) \to \left[ -\frac{1}{8}, \frac{1}{8} \right] \subseteq \mathbb{R} \) is not a local homomorphism. For this note that, for example \( \frac{\pi}{8} + \text{mod} \frac{1}{8} = \frac{1}{4} \in \theta \left( \left[ -\frac{1}{8}, \frac{1}{8} \right] \right) \), but \( \theta^{-1} \left( \frac{2}{8} \right) + \theta^{-1} \left( \frac{5}{8} \right) = -\frac{1}{4} \notin \left[ -\frac{1}{8}, \frac{1}{8} \right] \). In Claim 4.2.3 we formulate a necessary and sufficient condition on a local homomorphism \( \theta \) in order for \( \theta^{-1} \) to be a local homomorphism.

**Claim 4.2.3.** Let \( \theta : X \subseteq G_1 \to G_2 \) be an injective local homomorphism between the groups \( G_1 \) and \( G_2 \). Then

(i) \( \theta^{-1} : \theta(X) \subseteq G_2 \to X \subseteq G_1 \) is a local homomorphism if and only if for all \( y_1, y_2 \in \theta(X) \) if \( y_1 \cdot y_2 \in \theta(X) \), then \( \theta^{-1}(y_1) \cdot \theta^{-1}(y_2) \in X \).

(ii) If there is \( X' \subseteq X \) such that \( X' \cdot X' \subseteq X \), then \( \theta \mid_{X'}^{-1} : \theta(X') \to X' \) is a local homomorphism.

**Proof.** (i) Let \( y_1, y_2, y_1 \cdot y_2 \in \theta(X) \) such that \( \theta^{-1}(y_1) \cdot \theta^{-1}(y_2) \in X \), then \( \theta(\theta^{-1}(y_1) \cdot \theta^{-1}(y_2)) = y_1 \cdot y_2 \iff \theta^{-1}(y_1 \cdot y_2) = \theta^{-1}(y_1) \cdot \theta^{-1}(y_2) \). The other direction is clear.

(ii) Let \( \theta(x_1), \theta(x_2), \theta(x_1) \cdot \theta(x_2) \in \theta(X'). \) Since \( X' \cdot X' \subseteq X \), \( \theta(x_1 \cdot x_2) = \theta(x_1) \cdot \theta(x_2) \), then \( \theta^{-1}(\theta(x_1)) \cdot \theta^{-1}(\theta(x_2)) = x_1 \cdot x_2 = \theta^{-1}(\theta(x_1) \cdot \theta(x_2)) \in \theta^{-1}(\theta(X')) = X' \).

**Remark 4.2.4.** Let \( G_1 \) and \( G_2 \) be two groups, and \( W \subseteq G_1 \). Let \( \theta : W \to G_2 \) be an injective local homomorphism. If there is \( W' \subseteq W \) such that \( W' \cdot W' \subseteq W \), then \( (\theta \mid_{W'})^{-1} : \theta(W') \to W' \subseteq G_1 \) is a local homomorphism.

**Proof.** Let \( x, y \in W' \). Since \( W' \cdot W' \subseteq W \), \( \theta(x \cdot y) = \theta(x) \theta(y) \), then \( \theta^{-1}(\theta(x)) \theta^{-1}(\theta(y)) = x \cdot y = \theta^{-1}(\theta(x) \theta(y)) \).

54
4.3 A local homomorphism with generic domain between a semialgebraically compact semialgebraic group over R and the R-points of an R-algebraic group

Theorem 4.3.1. Let $G$ be a definably compact definably connected group definable in $\mathbb{R}$. Then there are

(i) a Zariski-connected $R$-algebraic group $H$ such that $\dim (G) = \dim (H(R)) = \dim (H)$,

(ii) a definable $X \subseteq G$ such that $G^{00} \subseteq X$,

(iii) a definable homeomorphism $\phi : X \subseteq G \to \phi(X) \subseteq H(R)$ such that $\phi$ and $\phi^{-1}$ are local homomorphisms.

Proof. Let $D = R(\sqrt{-1})$ be the algebraic closure of $R$. By Proposition 4.1.2, there are a finite subset $A \subseteq R$ over which $G$ is defined, a definably connected group $H$ definable in $D$ over $A$, a group-generic point $b$ of $G$ over $A$, a group-generic point $a$ of $G$ over $Ab$, thus $c = a \cdot b$ is also group-generic of $G$ over $Ab$ (this by Claim 3.3.12), as well as points $a', b', c' = a' \cdot b' \in H(R)$ generic in $H(R)$ over $A$ with the properties given there. Let $k$ be the subfield generated by $A$.

As every definably connected group definable over $k$ in the algebraic closed field $D$ is definably isomorphic over $k$ to a Zariski-connected $k$-algebraic group ([7, 47]), we may assume that $H$ is such algebraic group. Moreover, by the conditions of the points $a, b, c, a', b', c'$ of Proposition 4.1.2, the dimension of $H$ as algebraic group is equal to the o-minimal dimensions $\dim (G)$ and $\dim (H(R))$.

Since $a$ and $a'$ are interalgebraic over $k$ in $R$ and $R$ is o-minimal, $a$ and $a'$ are interdefinable over $k$ in $R$, and similarly for $b, b'$ and $c, c'$. From now on, we work in $R$ and by definable we will mean $R$-definable.

By [21, Lemma 4.8(i)] (which holds for $R$ instead of $\mathbb{R}$), there are open $k$-definable neighbourhoods $U, V$ and $W$ in $G$ of $a, b, c$, respectively, and $U', V', W'$ in $H(R)$ of $a', b', c'$, respectively, and $k$-definable functions $f, g$, and $h$ such that $f(a) = a'$ and $f$
is a definable homeomorphism between $U$ and $U'$, $g(b) = b'$ and $g$ is a definable homeomorphism between $V$ and $V'$, and $h(c) = c'$ and $h$ is a definable homeomorphism between $W$ and $W'$.

Let

$$Z = \{(x, y) \in G \times G : x \in U, y \in V, x \cdot y \in W, f(x) \cdot g(y) = h(x \cdot y)\}.$$

Since $b$ is group-generic in $G$ over $k$ and $a$ is group-generic in $G$ over $k b$, Corollary 3.4.4 yields $(a, b)$ is group-generic in $G \times G$ over $k$. Thus, as $Z$ is $k$-definable and $(a, b) \in Z$, then $Z$ is generic in $G \times G$.

By Lemma 3.4.9, there are definable sets $A, B$ generic in $G$ such that $A \times B \subseteq Z$.

Claim 4.3.2. Let $X, Y$ definable sets generic in $G$. Then there is $g \in G$ such that $X \cap (Y \cdot g^{-1})$ is generic in $G$ and that $(X \cap (Y \cdot g^{-1})) \cdot g \subseteq Y$.

Proof. By genericity of $X$ in $G$, there are $g_1, \ldots, g_k \in G$ such that $G = \bigcup_{i \leq k} X \cdot g_i$, hence $Y = \bigcup_{i \leq k} (X \cdot g_i) \cap Y = \bigcup_{i \leq k} (X \cap (Y \cdot g_i^{-1})) \cdot g_i$. Since $Y$ is generic, there is $i \leq k$ such that $(X \cap (Y \cdot g_i^{-1})) \cdot g_i$ is generic in $G$. Thus, with $g = g_i$ we get the desired result.

Then by the above claim applied to $A^{-1}$ and $B$, there is $g \in G$ such that $(A^{-1} \cap (B \cdot g^{-1})) \cdot g$ is generic and contained in $B$. Then if $A' = A \cap (g \cdot B^{-1})$, so $A'$ is generic in $G$ and $(A')^{-1} \cdot g \subseteq B$. By [3, Proposition 2.1], there is $s \in A'$ such that $s \cdot G^{00} \subseteq A'$. Since $A' = A \cap (g \cdot B^{-1})$, $s = g \cdot b^{-1}$ for some $b \in B$. Let $t = b$. Note that $s \cdot t \in A' \cdot B \subseteq W$. So far we have shown that there are generic sets $A'$ and $B$ in $G$ such that

(i) $A' \times B \subseteq Z$, and

(ii) there is $(s, t) \in A' \times B$ such that $G^{00} \cdot s^{-1} \subseteq (A')^{-1}$ and $(A')^{-1} \cdot (s \cdot t) \subseteq B$.

Let $X = (A')^{-1} \cdot s$, then $G^{00} \subseteq X$. Finally, we will define the local homomorphism. Proposition 4.3.3. The definable homeomorphism

$$\phi : X = (A')^{-1} \cdot s \to (f(A'))^{-1} \cdot f(s)$$
defined by \( \phi(x^{-1} \cdot s) = f(x)^{-1} \cdot f(s) \) for \( x \in A' \), and its inverse

\[
\phi^{-1} : (f(A'))^{-1} \cdot f(s) \rightarrow X,
\]

which is given by \( \phi^{-1}(y^{-1} \cdot f(s)) = (f^{-1}(y))^{-1} \cdot s \) for \( y \in f(A') \), are local homomorphisms between \( G \) and \( H(R)^0 \).

**Proof.** First, note the following.

**Claim 4.3.4.**

(i) If \( (x_1^{-1} \cdot s) \cdot (x_2^{-1} \cdot s) \in (A')^{-1} \cdot s \), then

\[
\phi \left( (x_1^{-1} \cdot s) \cdot (x_2^{-1} \cdot s) \right) = \phi(x_1^{-1} \cdot s) \cdot \phi(x_2^{-1} \cdot s) \Leftrightarrow
\]

\[
f \left( (x_1^{-1} \cdot s \cdot x_2^{-1})^{-1} \right) \cdot f(s) = f(x_1)^{-1} \cdot f(s) \cdot f(x_2)^{-1} \cdot f(s) \Leftrightarrow
\]

\[
f(x_1) f \left( (x_1^{-1} \cdot s \cdot x_2^{-1})^{-1} \right)^{-1} f(x_2) = f(s).
\]

(ii) If \( f(x_1)^{-1} \cdot f(s) \cdot (f(x_2)^{-1} \cdot f(s)) \in f(Y)^{-1} \cdot f(s) \), then

\[
\phi^{-1} \left( (f(x_1)^{-1} \cdot f(s)) \cdot (f(x_2)^{-1} \cdot f(s)) \right) = \phi^{-1}(f(x_1)^{-1} \cdot f(s)) \cdot \phi^{-1}(f(x_2)^{-1} \cdot f(s)) \Leftrightarrow
\]

\[
\left( f^{-1} \left( (f(x_1)^{-1} \cdot f(s) \cdot f(x_2)^{-1})^{-1} \right) \right)^{-1} \cdot s = x_1^{-1} \cdot s \cdot x_2^{-1} \cdot s \Leftrightarrow
\]

\[
x_1 \cdot \left( f^{-1} \left( (f(x_1)^{-1} \cdot f(s) \cdot f(x_2)^{-1})^{-1} \right) \right)^{-1} \cdot x_2 = s.
\]

**Proof.** It follows directly from the definitions of \( \phi \) and \( \phi^{-1} \).

\(\square\)

We will show in Claim 4.3.6 that each of \( \phi \) and \( \phi^{-1} \) satisfy one of the equivalent conditions formulated above. To prove Claim 4.3.6 we will use the next technical fact.

From now on, let \( s' = f(s) \) and \( t' = g(t) \).

**Claim 4.3.5.**

(i) For every \( y_1 \in f(A') \) and every \( y_2 \in g(B) \), \( f^{-1}(y_1) \cdot g^{-1}(y_2) = h^{-1}(y_1 \cdot y_2) \).

(ii) \( (f(A'))^{-1} \cdot (s' \cdot t') \subseteq g(B) \).
Proof. (i) Let $y_1 \in f(A')$, so there is $x_1 \in A'$ such that $f(x_1) = y_1$. Let $y_2 \in g(B)$, so there is $x_2 \in B$ such that $g(x_2) = y_2$. Since $A' \times B \subseteq Z$, then $y_1 \cdot y_2 = f(x_1) \cdot g(x_2) = h(x_1 \cdot x_2)$; therefore, $h^{-1}(y_1 \cdot y_2) = x_1 \cdot x_2 = f^{-1}(y_1) \cdot g^{-1}(y_2)$.

(ii) Let $x \in A'$, then $x^{-1} \cdot (s \cdot t) \in B$, so $g(x^{-1} \cdot (s \cdot t)) = (f(x))^{-1} \cdot h(s \cdot t) = (f(x))^{-1} \cdot (s' \cdot t')$. Hence, $(f(A'))^{-1} \cdot (s' \cdot t') \subseteq g(B)$. □

Claim 4.3.6. Let $x_1, x_2 \in A'$.

(i) Let $z^{-1} = x_1^{-1} \cdot s \cdot x_2^{-1}$. If $z^{-1} \in (A')^{-1}$, then $f(x_1) \cdot f(z)^{-1} \cdot f(x_2) = f(s) = s'$. (ii) Let $w^{-1} = f(x_1)^{-1} \cdot s' \cdot f(x_2)^{-1}$. If $w^{-1} \in f(A')^{-1}$, then $x_1 \cdot (f^{-1}(w))^{-1} \cdot x_2 = s = f^{-1}(s')$.

Proof. For the following recall that $(s,t) \in A' \times B$, and $A' \times B \subseteq Z$, so for every $(x,y) \in A' \times B$, $f(x) \cdot g(y) = h(x \cdot y)$.

(i)

$$f(s) \cdot g(t) = h(s \cdot t)$$
$$= h\left( (x_1 \cdot z^{-1} \cdot x_2) \cdot t \right)$$
$$= h\left( x_1 \cdot (z^{-1} \cdot x_2 \cdot t) \right)$$, since $z^{-1} \cdot x_2 \cdot t = x_1^{-1} \cdot s \cdot t \in (A')^{-1} \cdot s \cdot t \subseteq B$,
$$= f(x_1) \cdot g(z^{-1} \cdot x_2 \cdot t)$$
$$= f(x_1) \cdot f(z)^{-1} \cdot h(x_2 \cdot t)$$
$$= f(x_1) \cdot f(z)^{-1} \cdot f(x_2) \cdot g(t).$$

After cancelling $g(t)$, the desired conclusion is obtained.

(ii) By Claim 4.3.5, we have the following equations.

$$f^{-1}(s') \cdot g^{-1}(t') = h^{-1}(s' \cdot t')$$
$$= h^{-1}\left( (f(x_1) \cdot w^{-1} \cdot f(x_2)) \cdot t' \right)$$
$$= h^{-1}\left( f(x_1) \cdot (w^{-1} \cdot f(x_2) \cdot t') \right).$$

Since $w^{-1} \cdot f(x_2) \cdot t' = f(x_1)^{-1} \cdot s' \cdot t' \in g(B)$, then

$$h^{-1}\left( f(x_1) \cdot (w^{-1} \cdot f(x_2) \cdot t') \right) = f^{-1}(f(x_1)) \cdot g^{-1}(w^{-1} \cdot f(x_2) \cdot t'),$$

58
\[ f^{-1}(s') \cdot g^{-1}(t') = f^{-1}(f(x_1)) \cdot g^{-1}(w^{-1} \cdot f(x_2) \cdot t') = x_1 \cdot (f^{-1}(w))^{-1} \cdot h^{-1}(f(x_2) \cdot t') = x_1 \cdot (f^{-1}(w))^{-1} \cdot x_2 \cdot g^{-1}(t'). \]

After cancelling \( g^{-1}(t') \), we conclude the claim. \( \square \)

This finishes the proof of Proposition 4.3.3. \( \square \)

Theorem 4.3.1 is proved. \( \square \)

Claim 4.3.7. Let \( G \) be a definably connected definably compact group definable in a sufficiently saturated o-minimal expansion of a real closed field. Let \( X \subseteq G \) definable with \( G^{00} \subseteq X \). Then there are definable sets \( X_1, X_2 \subseteq G \) such that \( X_1 \) is definably simply connected, \( X_2 \) is definably connected and symmetric, and \( G^{00} \subseteq X_1 \subseteq X_2 \subseteq X \).

Proof. By saturation, \( G^{00} = \bigcap_{i \in \mathbb{N}} X_i = \bigcap_{i \in \mathbb{N}} X_i \cdot X_i^{-1} \), and since \( G^{00} \subseteq X \), then there is \( i \in \mathbb{N} \) such that \( G^{00} \subseteq X_i \cdot X_i^{-1} \subseteq X \). By the Cell decomposition Theorem ([48]), \( X_i \) is a finite union of definably simply connected cells, thus one of them has to be generic, call it \( C \). By [3, Proposition 2.1], there is \( g \in G \) such that \( G^{00} \subseteq C \cdot g \subseteq C \cdot C^{-1} \subseteq X_i \cdot X_i^{-1} \subseteq X \). Finally, let \( X_1 = C \cdot g \) and \( X_2 = C \cdot C^{-1} \). \( \square \)

Remark 4.3.8. By Claim 4.3.7, the definable generic set \( X \) of Theorem 4.3.1 can be taken either definably connected, symmetric, and \( G^{00} \subseteq X \), or definably simply connected and \( G^{00} \subseteq X \).

Let \( X \) be the set in the conclusion of Theorem 4.3.1. As \( G^{00} \subseteq X \), then there is \( X' \subseteq X \) symmetric such that \( G^{00} \subseteq X' \subseteq X' \cdot X' \subseteq X \). Then the next result holds in \( \mathcal{R} \):

Corollary 4.3.9. Let \( G \) be a definably compact definably connected group definable in \( \mathcal{R} \). Then there are

\( (i) \) a Zariski-connected \( R \)-algebraic group \( H \) such that \( \dim(G) = \dim(H(R)) = \dim(H) \),
(ii) definably connected definable sets $X', X \subseteq G$ such that $X'$ is symmetric and generic in $G$, and $X' \cdot X' \subseteq X$,

(iii) a definable homeomorphism $\phi : X \subseteq G \rightarrow \phi(X) \subseteq H(R)$ such that $\phi$ and $\phi^{-1}$ are local homomorphisms.

By transferring Corollary 4.3.9 from $\mathcal{R}$ to any real closed field, we have that Corollary 4.3.9 holds in any real closed field, not necessarily sufficiently saturated.
Chapter 5

Locally definable covering homomorphisms of locally definable groups

In this chapter, unless stated otherwise, we work in a sufficiently saturated o-minimal expansion of a real closed field $R$.

In this chapter we prove one of the main results of this work: the o-minimal universal covering group of an abelian connected definably compact group definable in a sufficiently saturated real closed field $R$ is, up to locally definable isomorphisms, an open connected locally definable subgroup of the o-minimal universal covering group of the $R$-points of some $R$-algebraic group (Theorem 5.6.2).

To prove this we study ld-spaces and their covering maps (Sect. 5.1), the notion and properties of the o-minimal universal covering homomorphism of locally definable groups (Sect. 5.2) along with some properties of definably generated groups in relation with generic sets and convex sets (Sect. 5.3). We also need to develop some results on local homomorphisms and their extensions to locally definable homomorphisms (Sections 5.4 - 5.6).
5.1 Ld-spaces and ld-covering maps

In [2] Baro and Otero introduced the locally definable category, which extends the locally semialgebraic one introduced by Delfs and Knebusch in [10] and is more flexible than the $\bigvee$-definable group category. $\bigvee$-definable groups are examples of locally definable spaces and their locally definable covering homomorphisms are locally definable covering maps of locally definable spaces. In what follows, we will introduce some definitions of the locally definable category from [2]. We assume at the beginning of this chapter that we work in a sufficiently saturated o-minimal expansion of a real closed field because our main results (e.g., Theorems 5.5.1, 5.6.1, and 5.6.2) are true under this assumption. However, we should say that Baro and Otero in [2] worked in an o-minimal expansion of a real closed field, not necessarily sufficiently saturated.

5.1.1 Ld-spaces and ld-maps

**Definition 5.1.1.** Let $M$ be a set. A locally definable space is a triple $(M, (M_i, \phi_i)_{i \in I})$ where

1. $M_i \subseteq M$, $M = \bigcup_{i \in I} M_i$, and $\phi_i : M_i \to Z_i$ is a bijection between $M_i$ and a definable set $Z_i \subseteq R^{m(i)}$ for every $i \in I$.
2. $\phi_i (M_i \cap M_j)$ is a definable relative open subset of $Z_i$ and the transition maps $\phi_{ij} = \phi_j \circ \phi^{-1}_i : \phi_i(M_i \cap M_j) \to M_i \cap M_j \to \phi_j(M_i \cap M_j)$ are definable for every $i, j \in I$.

The dimension of $M$ is $\dim (M) := \sup \{\dim (Z_i) : i \in I\}$. If $Z_i$ and $\phi_{ij}$ are definable over $A \subseteq R$ for all $i, j \in I$, we say that $M$ is a locally definable space over $A$.

Note that every definable space ([48, Chapter 10]) is a locally definable space with $|I| < \aleph_0$.

Every locally definable space $(M, (M_i, \phi_i)_{i \in I})$ has a unique topology on $M$ such that each $M_i$ is open and $\phi_i$ is a homeomorphism for all $i \in I$; more precisely,
$\mathcal{O} \subseteq M$ is open if and only if $\phi_i (\mathcal{O} \cap M_i)$ is relatively open in $Z_i$ for every $i \in I$. Throughout this subsection any topological property of locally definable spaces refers to this topology.

**Definition 5.1.2.** Let $(M, (M_i, \phi_i)_{i \in I})$ be a locally definable space.

(i) An ld-space is a Hausdorff locally definable space.

(ii) A subset $X \subseteq M$ is called a **definable subspace** of $M$ if there is a finite $J \subseteq I$ such that $X \subseteq \bigcup_{j \in J} M_j$ and $\phi_j (X \cap M_j)$ is definable for all $j \in J$.

(iii) A subset $Y \subseteq M$ is called a **compatible subspace** of $M$ if $\phi_i (Y \cap M_i)$ is definable for every $i \in I$, or equivalently, $Y \cap X$ is a definable subspace of $M$ for every definable subspace $X$ of $M$.

By Theorem 3.9 of [2], every $\bigvee$-definable group $U$ with its $\tau$-topology (see Fact 2.2.7) is an ld-space of finite dimension, and any definable subset of $U$ is a definable subspace of $U$.

We recall that any compatible subspace $Y$ of an ld-space $M$ inherits a natural structure of ld-space [2, Remark 2.3] given by $(Y, Y_i = Y \cap M_i, \phi_i |_{Y_i})$. And if $Y$ is a definable subspace then it inherits the structure of a definable space. Note that the only compatible subspaces of a definable space are the definable ones.

Now, we will introduce the maps between ld-spaces as in [2]. For this we note that given two ld-spaces $(M, (M_i, \phi_i)_{i \in I})$ and $(N, (N_j, \psi_j)_{j \in J})$ we can endow $M \times N$ with the structure $(M \times N, (M_i \times N_j, (\phi_i, \psi_j))_{i \in I})$ that makes it into an ld-space, and as it is defined in [48], a map $f : M \to N$ between definable spaces $M, N$ is a **definable map** if its graph is a definable subspace of $M \times N$.

**Definition 5.1.3.** A map $\theta : M \to N$ between ld-spaces (locally definable spaces) $(M, (M_i, \phi_i)_{i \in I})$ and $(N, (N_j, \psi_j)_{j \in J})$ is called an **ld-map** (locally definable map) if $\theta (M_i)$ is a definable subspace of $N$ and $\theta |_{M_i} : M_i \to \theta (M_i)$ is definable for every $i \in I$.

### 5.1.2 Some topological notions in ld-spaces

**Definition 5.1.4.** Let $M$ be an ld-space.
(i) $M$ is connected if $M$ has no compatible nonempty proper clopen subspace.

(ii) An ld-path in $M$ is a continuous ld-map $\alpha : [0, 1] \to M$, where $[0, 1] \subseteq R$ is considered as an ld-space with the trivial structure $([0, 1], ([0, 1], \text{id}))$ where $\text{id}$ is the identity function.

(iii) $M$ is path connected if for every $x_1, x_2 \in M$ there is an ld-path $\alpha : [0, 1] \to M$ such that $\alpha(0) = x_1$ and $\alpha(1) = x_2$.

(iv) The path connected component of a point $x \in M$ is the set of all $y \in M$ such that there is an ld-path from $x$ to $y$.

The next fact outlines some properties of connected ld-spaces. These are Remarks 4.1 and 4.3, and Fact 4.2 of [2].

**Fact 5.1.5.** Let $M$ be an ld-space.

(i) $M$ is connected if and only if $M$ is path connected if and only if every ld-map from $M$ to a discrete ld-space is constant.

(ii) Every path connected component of an ld-space is a clopen compatible subspace.

**Claim 5.1.6.** Let $M = \bigcup_{i \in I} X_i$ be an ld-space such that $\{X_i : i \in I\}$ is a collection of connected compatible subspaces of $M$ and $\bigcap_{i \in I} X_i \neq \emptyset$. Then $M$ is connected.

**Proof.** Let $Y \subseteq M$ be a compatible nonempty clopen in $M$. Since $Y \neq \emptyset$, there is $k \in I$ such that $Y \cap X_k \neq \emptyset$. Since $X_k$ and $Y$ are compatible in $M$, so is $Y \cap X_k$, and in particular $Y \cap X_k \subseteq X_k$ is a clopen compatible set in $X_k$. By the connectedness of $X_k$, $Y \cap X_k = X_k$.

As $\bigcap_{i \in I} X_i \neq \emptyset$, $X_i \cap X_k \neq \emptyset$ for every $i \in I$, then $X_i \cap Y \neq \emptyset$, and as above we conclude that $Y \cap X_i = X_i$ for every $i \in I$. Therefore, $M = \bigcup_{i \in I} X_i = \bigcup_{i \in I} Y \cap X_i = Y$. Thus $M$ has no clopen proper nonempty compatible subset.

**Corollary 5.1.7.** Let $M, N$ be two connected ld-spaces. Then the product ld-space $M \times N$ is connected.
Proof. Fix $y \in N$. For $x \in M$, let $T_x = (\{x\} \times N) \cup (M \times \{y\})$. Since $(x, y) \in (\{x\} \times N) \cap (M \times \{y\})$, Claim 5.1.6 implies that $T_x$ is connected. Finally, as $\bigcap_{x \in M} T_x = M \times \{y\}$, again Claim 5.1.6 implies that $\bigcup_{x \in M} T_x = M \times N$ is connected. \hfill \Box

**Proposition 5.1.8.** Let $\mathcal{U}$ be a locally definable group and $X \subseteq \mathcal{U}$ a connected definable set such that the identity element $e_\mathcal{U} \in X$. Then the subgroup $\langle X \rangle$ of $\mathcal{U}$ generated by $X$ is a connected locally definable group.

**Proof.** By Corollary 5.1.7, $\underbrace{X \times \cdots \times X}_{i\text{-times}}$ is a connected definable space for every $i \in \mathbb{N}$. Since the ld-map

$$
p_i : \langle X \rangle \times \cdots \times \langle X \rangle \rightarrow \langle X \rangle \\
(x_1, \ldots, x_i) \mapsto \prod_{1 \leq j \leq i} x_j
$$

is continuous (with respect to their topologies of locally definable groups) and the image of a connected ld-space by a continuous ld-map is connected, then $p_i \left( \underbrace{X \times \cdots \times X}_{i\text{-times}} \right) = \prod_i X$ is connected.

Finally, as $\bigcap_{i \in \mathbb{N}} \prod_i (X \cup X^{-1}) \supseteq (X \cup X^{-1}) \neq \emptyset$, then $\bigcup_{i \in \mathbb{N}} \prod_i (X \cup X^{-1}) = \langle X \rangle$ is connected by Claim 5.1.6. \hfill \Box

**Definition 5.1.9.** Let $M$ be an ld-space and $x_0 \in M$. Let $\alpha, \gamma : [0, 1] \rightarrow M$ be two ld-paths. A continuous ld-map $H(t, s) : [0, 1] \times [0, 1] \rightarrow M$ is a *homotopy* between $\alpha$ and $\gamma$ if $\alpha = H(\cdot, 0)$ and $\gamma = H(\cdot, 1)$, where $[0, 1] \times [0, 1] \subseteq R \times R$ is considered as an ld-space with the trivial structure $([0, 1] \times [0, 1], ([0, 1] \times [0, 1], (id, id)))$ where id is the identity function. In this case, $\alpha$ and $\gamma$ are called *homotopic*, denoted $\alpha \sim \gamma$.

Let $\mathbb{L}(M, x_0)$ be the set of all ld-paths that start and end at the element $x_0 \in M$. Note that being homotopic $\sim$ is an equivalence relation on $\mathbb{L}(M, x_0)$. We define the *o-minimal fundamental group* $\pi_1(M, x_0) := \mathbb{L}(M, x_0) / \sim$. Observe that $\pi_1(M, x_0)$ is a group with the operation given by the class of the concatenation of its representatives; i.e., $[\alpha] \cdot [\gamma] = [\alpha \cdot \gamma]$. In case $M$ is a connected locally definable group, $\pi_1(M, x_0)$ is an abelian group ([11, Prop. 4.1]).

$M$ is called *simply connected* if $M$ is path connected and $\pi_1(M, x_0)$ is the trivial group.
5.1.3 Covering maps for ld-spaces

The next definition of covering map for ld-spaces is taken from [2].

**Definition 5.1.10.** Let \((M, (M_i, \phi_i)_{i \in I})\) and \((N, (N_j, \psi_j)_{j \in J})\) be ld-spaces. A surjective continuous ld-map \(\theta : M \to N\) is called an *ld-covering map* if there is a family \(\{O_l : l \in L\}\) of open definable subspaces of \(N\) such that

(i) \(N = \bigcup_{l \in L} O_l\), and

(ii) for every \(l \in L\) and each connected component \(C\) of \(\theta^{-1}(O_l)\), the restriction \(\theta|_C : C \to O_l\) is a definable homeomorphism (so in particular both \(C\) and \(\theta|_C\) are definable).

We call \(\{O_l : l \in L\}\) a *\(\theta\)-admissible family* of definable neighborhoods.

**Remark 5.1.11.** Let \((M, (M_i, \phi_i)_{i \in I})\) and \((N, (N_j, \psi_j)_{j \in J})\) be ld-spaces, and let \(\theta : M \to N\) be a surjective continuous ld-map. Then \(\theta : M \to N\) is an ld-covering map if and only if there is a family \(\{O_l : l \in L\}\) of open definable subspaces of \(N\) such that

(i) \(N = \bigcup_{l \in L} O_l\), and

(ii) for every \(l \in L\), \(\theta^{-1}(O_l)\) is a disjoint union \(\bigcup_{i \in L_l} O'_{l,i}\) of open definable subspaces of \(M\) such that for every \(i \in L_l\) the restriction \(\theta|_{O'_{l,i}} : O'_{l,i} \to O_l\) is a definable homeomorphism (so in particular both \(O'_{l,i}\) and \(\theta|_{O'_{l,i}}\) are definable).

**Proof.** (\(\Rightarrow\)) This direction is clear since we can take the same \(\theta\)-admissible family \(\{O_l : l \in L\}\) given by hypothesis, and note that the family of (path) connected components of the open compatible subspace \(\theta^{-1}(O_l)\) is a family of open definable subspaces of \(M\) each of which is definably homeomorphic to \(O_l\).

(\(\Leftarrow\)) Let \(\{O_l : l \in L\}\) be the family of open definable subspaces of \(N\) given by hypothesis where \(\theta^{-1}(O_l) = \bigcup_{i \in L_l} O'_{l,i}\) for every \(l \in L\). Then

\[
\{\theta(C) : C \text{ is a connected component of } O'_{l,i}, i \in L_l, l \in L\}
\]
is a \(\theta\)-admissible family of definable neighborhoods.

Now, we will prove that any ld-covering map between ld-spaces is closed for definable subspaces; notice that such a map is always open.

**Remark 5.1.12.** Let \((M, (M_i, \phi_i)_{i \in I})\) be an ld-space and \(X \subseteq M\) a definable space. If \(y \in \text{Cl}(X)\), then there is a definable map \(g : (0, \epsilon) \to X\), for some \(\epsilon > 0\), such that \(\lim_{t \to 0} g(t) = y\).

**Proof.** Since \(X\) is a definable subspace of \(M\), then there is a finite \(J \subseteq I\) such that \(X \subseteq \bigcup_{j \in J} M_j\). As \(y \in \text{Cl}(X)\), then \(y \in \text{Cl}(X \cap M_j)\) for some \(j \in J\). Also, since \(y \in M\), there is \(k \in I\) such that \(y \in M_k\). Since \(M_k\) is open in \(M\) and \(y \in \text{Cl}(X \cap M_j)\), then \(M_k \cap X \cap M_j \neq \emptyset\).

Because the definable spaces of an ld-space are closed under finite intersections, then \(\phi_k(M_k \cap X \cap M_j)\) is a definable set in \(Z_k\). As \(y \in \text{Cl}(M_k \cap X \cap M_j)\) and \(\phi_k\) is a homeomorphism, then \(\phi_k(y) \in \text{Cl}(\phi_k(M_k \cap X \cap M_j))\). Then, by [29, Thm. 4.8], there is a definable map \(\gamma : (0, \epsilon) \to \phi_k(M_k \cap X \cap M_j)\) such that \(\lim_{t \to 0} \gamma(t) = \phi_k(y)\). Because \(\phi_k\) is a homeomorphism between \(M_k\) and \(Z_k\), \(g = \phi_k^{-1} \circ \gamma : (0, \epsilon) \to M_k \cap X \cap M_j\) is a definable map such that \(\lim_{t \to 0} g(t) = y\). □

**Proposition 5.1.13.** Let \((M, (M_i, \phi_i)_{i \in I}), (N, (N_j, \psi_j)_{j \in J})\) be ld-spaces, a definable subspace \(C \subseteq M\) closed in \(M\), and \(\theta : M \to N\) an ld-covering map. Then \(\theta(C) \subseteq N\) is a definable subspace closed in \(N\).

**Proof.** We will show that if \(y \in \text{Cl}(\theta(C))\), then \(y \in \theta(C)\). As the image of a definable space by an ld-map is a definable space, \(\theta(C)\) is a definable space. Because \(y \in \text{Cl}(\theta(C))\), Remark 5.1.12 yields the existence of a definable map \(g : (0, \epsilon) \to \theta(C)\) such that \(\lim_{t \to 0} g(t) = y\).

Now, since \(\theta : M \to N\) is an ld-covering map, there is an open definable subspace \(\mathcal{O} \subseteq N\) such that \(y \in \mathcal{O}\), \(\theta^{-1}(\mathcal{O}) = \bigcup_{j \in J} \mathcal{O}_j\), and each \(\mathcal{O}_j\) is an open definable subspace in \(M\) homeomorphic to \(\mathcal{O}\) by \(\theta\).

Since \(\lim_{t \to 0} g(t) = y\) and \(\mathcal{O}\) is an open neighborhood of \(y\), there is \(\delta > 0\) such that \(\delta \leq \epsilon\) and \(g((0, \delta)) \subseteq \mathcal{O}\). Without loss of generality, we can assume that \(\text{dom}(g) = (0, \delta)\).
Let \( C' = (\theta |_C)^{-1} (g((0, \delta))) \). Since \( C' \subseteq \theta^{-1}(\mathcal{O}) = \bigcup_{j \in S} \mathcal{O}'_{i_j} \) and \( C' \) is definable, then saturation implies that \( C' \subseteq \mathcal{O}'_{i_1} \cup \ldots \cup \mathcal{O}'_{i_k} \) for some \( k \in \mathbb{N} \). As \( (0, \delta) = g^{-1}(\theta(C')) \), then \( (0, \delta) = g^{-1}(\theta(C')) = g^{-1}(\bigcup_{1 \leq i \leq k} \theta(\mathcal{O}'_{i_j} \cap C')) = \bigcup_{1 \leq i \leq k} g^{-1} \circ \theta(\mathcal{O}'_{i_j} \cap C') \). Since \( \theta |_{\mathcal{O}'}: \mathcal{O}_j' \to \mathcal{O} \) is a definable homeomorphism for every \( j \in S \), any path \( f \) in \( \mathcal{O} \) can be lifted through \( \theta |_{\mathcal{O}_j'} \) to a path \( \tilde{f} \) in \( \mathcal{O}_j' \). Therefore, in particular, for each \( i \in \{1, \ldots, k\} \), \( \tilde{g}_{j_i} = \theta |_{\mathcal{O}_j'}^{-1} \circ g \) is a definable path in \( \mathcal{O}_j' \cap C' \). Hence, \( (0, \delta) = \bigcup_{1 \leq i \leq k} \tilde{g}_{j_i}^{-1} (\mathcal{O}_j' \cap C') \).

By \( \alpha \)-minimality, there are \( j \in \{1, \ldots, k\} \) and a positive \( \epsilon^* < \delta \) such that \( (0, \epsilon^*) \subseteq \tilde{g}_{j_i}^{-1} (\mathcal{O}_j' \cap C') \). Let \( x = \theta |_{\mathcal{O}_j'}^{-1} (y) \). Since \( g |_{(0, \epsilon^*)} (t) \to y \), then the lifting \( \tilde{g}_{j_i} |_{(0, \epsilon^*)} (t) \to x \). So, \( x \in \text{Cl}(C) \). But \( C \) is closed in \( \mathcal{U} \), so \( x \in C \); namely, \( y \in \theta(C) \). Then the image by \( \theta \) of any definable closed subspace of \( M \) is closed in \( N \). This ends the proof of Proposition 5.1.13.

With the previous proposition we can prove the existence of a homeomorphism between simply connected definable spaces as a restriction of a given ld-covering map as we see in the following proposition.

**Proposition 5.1.14.** Let \( M, N \) be ld-spaces and \( \theta : M \to N \) an ld-covering map. Let \( Y \subseteq N \) be a compatible subspace in \( N \), then \( \theta |_{\theta^{-1}(Y)}: \theta^{-1}(Y) \to Y \) is an ld-covering map of ld-spaces. If, moreover, \( Y \) is definable, \( n_0 \in Y \), and \( m_0 \in \theta^{-1}(n_0) \), then there is a definable subspace \( W \subseteq M \) open in \( \theta^{-1}(Y) \) such that \( m_0 \in W \) and the following hold.

(i) \( \theta |_W: W \to Y \) is a definable covering map of ld-spaces.

(ii) If in addition \( Y \) is simply connected, then \( \theta |_{W_{m_0}}: W_{m_0} \to Y \) is a homeomorphism of definable spaces where \( W_{m_0} \) is the connected component of \( m_0 \) in \( W \).

**Proof.** Since the preimage of a compatible subspace by an ld-map is a compatible subspace, \( \theta^{-1}(Y) \) is a compatible subset of \( M \). As \( \theta |_{\theta^{-1}(Y)}: \theta^{-1}(Y) \to Y \) is a continuous surjection, it only remains to show the existence of a \( \theta |_{\theta^{-1}(Y)} \)-admissible family of definable neighborhoods. Let \( \{\mathcal{O}_i\}_{i \in L} \) be a \( \theta \)-admissible family of definable neighborhoods such that \( \theta^{-1}(\mathcal{O}_i) = \bigcup_{j \in S_i} \mathcal{O}'_{i_j} \) and \( \mathcal{O}'_{i_j} \) is ld-homeomorphic to \( \mathcal{O}_i \) by \( \theta \) for any \( i \in L, j \in S_j \). Then \( \{\mathcal{O}_i \cap Y\}_{i \in L} \) is a \( \theta |_{\theta^{-1}(Y)} \)-admissible family of definable neighborhoods of \( Y \) in \( M \). Therefore, the image of any definable subspace of \( Y \) by \( \theta \) is definable and \( \theta^{-1}(Y) \) is a definable covering map of ld-spaces. This finishes the proof.

68
neighborhoods because for every \( i \in L \), \( O_i \cap Y \) is a definable subspace open in \( Y \), \( \theta \mid_{\theta^{-1}(Y)} (O_i \cap Y) = \bigcup_{j \in S_i} O'_{i_j} \cap \theta^{-1}(Y) = \bigcup_{j \in S_i} \theta \mid_{O'_{i_j}}^{-1} (O_i \cap Y) \) is a disjoint union of definable subspaces open in \( \theta^{-1}(Y) \), and each \( O'_{i_j} \cap \theta^{-1}(Y) \) is homeomorphic to \( O_i \cap Y \) by \( \theta \).

Now, assume that \( Y \) is also a definable space. In what follows, we will prove (i). Let \( \{O_i \cap Y\}_{i \in L} \) be the above \( \theta \mid_{\theta^{-1}(Y)} \)-admissible family of definable neighborhoods for \( \theta \mid_{\theta^{-1}(Y)} : \theta^{-1}(Y) \rightarrow Y \). Hence, the definability of \( Y \) and the saturation of the model imply that there is \( s \in \mathbb{N} \) such that \( Y = \bigcup_{1 \leq i \leq s} O_i \cap Y \). For each \( i \in \{1, \ldots, s\} \) fix an arbitrary finite nonempty subset \( S' \subseteq S_i \) such that if \( e_{m_0} \in O_i \cap Y \), then there is \( j \in S' \) such that \( e_{m_0} \in O'_{i_j} \cap \theta^{-1} (Y) \). Let

\[
W = \bigcup \left\{ O'_{i_j} \cap \theta^{-1}(Y) : i \in \{1, \ldots, s\}, j \in S' \right\},
\]

which is open in \( \theta^{-1}(Y) \). Then \( \{O_i \cap Y : i \in \{1, \ldots, s\}\} \) is a \( \theta \mid_W \)-admissible family of definable neighborhoods. So \( \theta \mid_W : W \rightarrow Y \) is a definable covering map of ld-spaces.

For (ii), first we will prove that \( \theta \mid_{W_{m_0}} : W_{m_0} \rightarrow Y \) is a definable covering map of ld-spaces, this is the next claim.

**Claim 5.1.15.** Let \( W_{m_0} \) be the connected component of \( m_0 \) in \( W \). Then

(i) \( \theta \mid_{W_{m_0}} : W_{m_0} \rightarrow Y \) is surjective.

(ii) There is a \( \theta \mid_{W_{m_0}} \)-admissible family of definable neighborhoods.

Therefore, \( \theta \mid_{W_{m_0}} : W_{m_0} \rightarrow Y \) is a definable covering map of ld-spaces.

**Proof.** (i) By Fact 4.2 of [2], \( W_{m_0} \) is a clopen definable subset of \( W \). By Proposition 5.1.13, \( \theta (W_{m_0}) \) is a definable space clopen in \( Y \), but \( Y \) is connected, so \( \theta (W_{m_0}) = Y \); i.e., \( \theta \) is surjective.

(ii) The same \( \theta \mid_W \)-admissible family of definable neighborhoods \( \{O_i \cap Y : i \in \{1, \ldots, s\}\} \) works for \( \theta \mid_{W_{m_0}} \) because if \( C \) is a connected component of \( \theta \mid_W^{-1} (O_i \cap Y) \) in \( W \), then \( C \) is either entirely contained in \( W_{m_0} \) or is disjoint from \( W_{m_0} \). Therefore, \( C \) is homeomorphic by \( \theta \mid_{W_{m_0}} \) with \( O_i \cap Y \).

69
From (i) and (ii), $\theta \mid_{W_{m_0}} : W_{m_0} \to Y$ is a definable covering map of ld-spaces, which ends the proof of Claim 5.1.15.

Since $Y$ is simply connected, [15, Remark 3.8] implies that there is an ld-covering map $\beta : Y \to W_{m_0}$ such that $\text{id} = \theta \mid_{W_{m_0}} \circ \beta$, then $\theta \mid_{W_{m_0}} : W_{m_0} \to Y$ is a definable homeomorphism. This concludes the proof of Proposition 5.1.14.

\[\square\]

5.2 The o-minimal universal covering homomorphism of a locally definable group

This section is devoted to introducing the notion and properties of locally definable covering homomorphism and o-minimal universal covering homomorphism.

**Definition 5.2.1.** Let $\mathcal{U}, \mathcal{V}$ be locally definable groups. An ld-covering map $\theta : \mathcal{U} \to \mathcal{V}$ that is also a homomorphism is called a **locally definable covering homomorphism**.

As before, $\{U_i\}_{i \in I}$ is called a $\theta$-admissible family of definable neighborhoods.

Two locally definable covering homomorphisms $\theta : \mathcal{U} \to \mathcal{V}, \theta' : \mathcal{U}' \to \mathcal{V}$ are called **equivalent** if there are locally definable covering homomorphisms $\beta : \mathcal{U} \to \mathcal{U}'$ and $\beta' : \mathcal{U}' \to \mathcal{U}$ such that $\theta = \theta' \circ \beta$ and $\theta' = \theta \circ \beta'$, so the following diagram commutes.

\[
\begin{array}{ccc}
\mathcal{U} & \xrightarrow{\beta} & \mathcal{U}' \\
\theta \downarrow & & \theta' \downarrow \\
\mathcal{V} & \xrightarrow{\beta'} & \mathcal{U}
\end{array}
\]

In general, in our diagrams the regular arrows are maps whose existence is assumed, and the dashed arrows are maps whose existence is asserted. The inclusion map is denoted by $i$.

**Fact 5.2.2.** [11, Theorem 3.6] Let $\theta : \mathcal{U} \to \mathcal{V}$ be a surjective locally definable homomorphism between locally definable groups. If $\ker(\theta)$ has dimension zero, then $\theta : \mathcal{U} \to \mathcal{V}$ is a locally definable covering homomorphism.

70
**Definition 5.2.3.** Let \( V \) be a connected locally definable group. A locally definable covering homomorphism \( \theta : U \to V \) with \( U \) connected is called an *o-minimal universal covering homomorphism* of \( V \) if for every locally definable covering homomorphism \( \pi : Z \to V \) with \( Z \) connected, there exists a locally definable covering homomorphism \( \beta : U \to Z \) such that \( \theta = \pi \circ \beta \). In this case \( Z \) is called an *o-minimal universal covering group* of \( V \).

Note that if there are two o-minimal universal covering homomorphisms \( \theta : U \to V \) and \( \theta' : U' \to V \) of a connected locally definable group \( V \), then there exist locally definable covering homomorphisms \( \beta : U \to U' \) and \( \beta' : U' \to U \) such that \( \theta = \theta' \circ \beta \) and \( \theta' = \theta \circ \beta' \). Therefore, if \( V \) has an o-minimal universal covering homomorphism, then it is unique up to equivalent locally definable covering homomorphisms. Thus, we can say “the” o-minimal universal covering homomorphism of \( V \), and sometimes we denote the o-minimal universal covering group of \( V \) by \( \tilde{V} \).

In [14] Edmundo and Eleftheriou constructed a locally definable covering homomorphism \( \theta : U \to V \) for a given connected locally definable group \( V \) that satisfies the definition of an o-minimal universal covering homomorphism of \( V \) (Def. 5.2.3) (so the o-minimal universal covering homomorphism of \( V \) exists), and they showed the following.

**Fact 5.2.4.** [14, Thm. 3.11] For a connected locally definable group \( V \), the kernel of its o-minimal universal covering homomorphism is isomorphic, as abstract groups, to the o-minimal fundamental group \( \pi_1(V) \).

**Fact 5.2.5.** [15, Remark 3.8] A locally definable covering homomorphism \( \pi : U \to V \) between connected locally definable groups \( U \) and \( V \) is the o-minimal universal covering homomorphism of \( V \) if and only if \( \pi_1(U) = \{0\} \).

**Remark 5.2.6.** Let \( U, V \) be connected locally definable groups, and \( \theta : U \to V \) a locally definable covering homomorphism. Then

(i) \( U \) is abelian if and only if \( V \) is abelian.

(ii) Assume that \( V \) is abelian. Then \( U \) is divisible if and only if \( V \) is divisible.

**Proof.** (i) Clearly, if \( U \) is abelian, by the surjectiveness of \( \theta \), \( V \) is abelian.
As in the category of topological groups, we can consider the space $V^*$ of homotopy classes of ld-paths in $V$ based at the identity and with end point some element in $V$, whose product of paths is given by pointwise multiplication and whose covering map sends each path class to its endpoint. In [14] Edmundo and Eleftheriou used this abstract universal covering group $V^*$ to obtain a locally definable covering homomorphism $\pi: \tilde{V} \to V$ which is precisely the o-minimal universal covering homomorphism of $V$; moreover, $V$ and $V^*$ are isomorphic, as abstract groups. Thus, since $V^*$ is abelian if and only if $V$ is abelian, then $\tilde{V}$ is abelian if and only if $V$ is abelian.

Now, assume that $V$ is abelian, and let $\pi: \tilde{V} \to V$ be the o-minimal universal covering homomorphism of $V$. Then there is a locally definable covering homomorphism $\beta: \tilde{V} \to U$ such that $\pi = \theta \circ \beta$. Since $V$ is abelian, so is $\tilde{V}$, then, by going through $\beta$, $U$ is also abelian.

(ii) It is clear that if $U$ is divisible, by the surjectiveness of $\theta$, $V$ is divisible. The other implication needs the abelianness of the groups, and it is [4, Proposition 5.13].

\[ \square \]

**Fact 5.2.7.** [4, Proposition 5.14] The o-minimal universal covering group of a connected abelian divisible locally definable group is divisible and torsion free.

**Claim 5.2.8.** Let $U$ be a connected locally definable group covering an abelian connected definable group $G$. If $U$ is torsion free, then $U$ is simply connected.

\begin{proof}
Since $G$ is an abelian (definably) connected definable group, then $G$ is divisible (see, e.g., the proof of [16, Theorem 2.1]). Then $U$ is also abelian and divisible, by Remark 5.2.6. So the map $p_k : U \to U : x \mapsto x^k$ is a bijective locally definable homomorphism for any $k \in \mathbb{N}$, so in particular $p_k$ is a locally definable covering homomorphism. Thus, by [2, Corollary 6.12] or [11, Proposition 4.6], the induced map $p_{k,*} : \pi_1 (U) \to \pi_1 (U) : [\gamma] \mapsto [p_k \circ \gamma]$ is an injective homomorphism; therefore, the $k$-torsion group $U [k]$ of $U$ satisfies that $\{0\} = U [k] = \ker (p_{k,*}) \cong \pi_1 (U) / p_{k,*} (\pi_1 (U))$. Then, $\pi_1 (U) = (\pi_1 (U))^k$ for every $k \in \mathbb{N}$, thus $\pi_1 (U)$ is a divisible group.

Now, let $\theta : U \to G$ be a locally definable covering homomorphism, and $\theta_\ast : \pi_1 (U) \to \pi_1 (G)$ its induced injective homomorphism, so $\theta_\ast (\pi_1 (U))$ is a divisible subgroup of $\pi_1 (G)$. By [16, Theorem 2.1], there is $s \in \mathbb{N}$ such that $\pi_1 (G) \cong \mathbb{Z}^s$, then the only possible divisible subgroup of $\pi_1 (G)$ is the trivial one, so $\pi_1 (U) = \{0\}$.

72
From Fact 5.2.7 and Claim 5.2.8, we have that if $G$ is a connected abelian definable group, $G$ is torsion free if and only if $G$ is simply connected.

**Corollary 5.2.9.** Let $U$ be a connected torsion free locally definable group, $G$ an abelian connected definable group, and $\theta : U \to G$ a locally definable covering homomorphism. Then $\theta : U \to G$ is the o-minimal universal covering homomorphism of $G$.

**Proof.** By [15, Remark 3.8] and Claim 5.2.8, $U$ is simply connected. So, by Fact 5.2.5, $\theta : U \to G$ is the o-minimal universal covering homomorphism of $G$. $\square$

### 5.3 Abelian definably generated groups, convex sets, and covers of definable groups

In this section we present some properties of the abelian $\bigvee$-definable groups in relation to their smallest type-definable subgroup of index smaller than $\kappa$, if it exists, and to some generic subsets and convex sets.

Note that if $U$ is a connected $\bigvee$-definable group with $U^{00}$, then $U$ has a definable left-generic set, thus, by Fact 2.3 in [17], $U$ is definably generated, and hence locally definable.

In the first part of this section, we point out some central facts about the existence of $U^{00}$ for an abelian definably generated group $U$ as well as necessary and sufficient conditions for being a cover of a definable group. The first of these facts gathers Proposition 3.5 and Theorem 3.9 of the work of Peterzil and Eleftheriou in [17].

**Fact 5.3.1.** [17] Let $U$ be a connected abelian definably generated group of dimension $d$. Then:

- (i) $U$ covers a definable group if and only if the subgroup $U^{00}$ exists if and only if $U$ contains a definable generic set.
(ii) If $U^{00}$ exists, then $U^{00}$ is torsion free, $U$ and $U^{00}$ are divisible, and $U/U^{00}$ is a Lie group isomorphic, as a topological group, to $\mathbb{R}^k \times T^r$ for some $k, r \in \mathbb{N}$ with $k + r \leq d$, where $T$ is the circle group.

**Definition 5.3.2.** [4, Def. 5.3] Let $G$ be an abelian group and $X \subseteq G$.

(i) $X$ is called convex if for every $a, b \in X$ and $n, m \in \mathbb{N}$, not both null, $X$ contains every solution $x \in G$ of the equation $x^{m+n} = a^m b^n$.

(ii) The convex hull $\text{ch}(X)$ of $X$ is the set of all $x \in G$ such that $x^n = a_1 \cdots a_n$ for some $n \in \mathbb{N}$ and some $a_1, \ldots, a_n \in X$ not necessarily distinct.

(iii) A locally definable abelian group $U$ has definably bounded convex hulls if for all definable $X \subseteq U$, there is a definable $Y \subseteq U$ such that $\text{ch}(X) \subseteq Y$.

If $U$ is a divisible torsion free abelian group, then it is easy to prove that $X \subseteq U$ is convex if and only if $\prod_n X = X^n$ for every $n \in \mathbb{N}$.

**Fact 5.3.3.** [4, Theorem 5.6] Let $U$ be a connected abelian definably generated group. The following are equivalent:

(i) $U$ covers a definable group.

(ii) For every definable $X \subseteq U$, there is a definable $Y \subseteq U$ such that $\prod_n X \subseteq Y^n$ for all $n \in \mathbb{N}$.

(iii) $U$ is divisible and has definably bounded convex hulls.

For the second part of this section, we first make some basic observations about type-definable groups.

**Remark 5.3.4.** Let $G = \bigcap_{i < \kappa} X_i$ be a type-definable subgroup of a $\bigvee$-definable group $U$ such that $\{X_i\}_{i < \kappa}$ is a decreasing sequence of definable subsets of $U$. Then for every $p \in \mathbb{N}$, $G = \bigcap_{i < \kappa} \prod_p X_i$.

**Proof.** Since the identity element of $U$ belongs to $X_i$ and $X_i \subseteq \prod_p X_i$ for every $i < \kappa$, then $G \subseteq \bigcap_{i < \kappa} \prod_p X_i$. For the another inclusion, we will see that for every $n < \kappa$
there is a finite set $I(n) = \{i_{j,k} : 1 \leq j \leq p, 1 \leq k \leq n\} \subseteq \mathbb{N}$ such that $\bigcap_{1 \leq k \leq n} X_{n,k} \cdot \bigcap_{1 \leq k \leq n} X_{i_{2,k}} \cdots \bigcap_{1 \leq k \leq n} X_{i_{p,k}} \subseteq X_n$. Suppose towards a contradiction that such set $I(n)$ does not exist. Then saturation implies that $\left(\prod_p \bigcap_{i \in \kappa} X_i\right) \cap X_n \neq \emptyset$, but this contradicts that $G = \prod_p G \subseteq X_n$ for every $n < \kappa$. Then such finite set $I(n)$ exists.

Now, since $\{X_i\}_{i<\kappa}$ is a decreasing sequence, then $\prod_p X_{i(n)^*} \subseteq X_n$ where $i(n)^* = \max \{i : i \in I(n)\}$. Therefore, $\bigcap_{i<\kappa} \prod_p X_i \subseteq \bigcap_{n<\kappa} X_n = G$. This ends the proof of the claim.

**Remark 5.3.5.** In the same way, it can be shown that any type-definable subgroup of a $\bigvee$-definable group is the intersection of less than $\kappa$-many symmetric definable subsets. Then, by Remark 5.3.4, for every type-definable subgroup $G$ of a $\bigvee$-definable group $U$ there is a decreasing sequence $\{X_i\}_{i<\kappa}$ of symmetric definable subsets of $U$ such that for every $p \in \mathbb{N}$, $G = \bigcap_{i<\kappa} \prod_p X_i$.

**Claim 5.3.6.** Let $L$ be a topological group isomorphic, as a topological group, to $\mathbb{R}^k \times \mathbb{T}^r$ for some $k, r \in \mathbb{N}$, where $\mathbb{T}$ is the circle group. Let $C \subseteq L$ be a compact neighbourhood of the identity element $e_L$ of $L$. Then there is an increasing sequence $\{n_i\}_i \subseteq \mathbb{N}$ such that $C^{n_i} \subseteq C^{n_{i+1}}$ for every $i \in \mathbb{N}$, and $L = \bigcup_{i \in \mathbb{N}} C^{n_i}$.

**Proof.** First, note that in $\mathbb{R}^k \times \mathbb{T}^r$ every compact neighbourhood $Y \subseteq \mathbb{R}^k \times \mathbb{T}^r$ of the identity element $e$ of $\mathbb{R}^k \times \mathbb{T}^r$, there is a neighbourhood $X \subseteq Y$ of $e$ such that $\mathbb{R}^k \times \mathbb{T}^r = \bigcup_{n \in \mathbb{N}} X^n$ and $X^n \subseteq X^{n+1}$. Therefore, as $L$ and $\mathbb{R}^k \times \mathbb{T}^r$ are isomorphic as a topological groups, then there is a neighbourhood $O \subseteq C$ of $e_L$ such that $L = \bigcup_{n \in \mathbb{N}} O^n$, and $O^n \subseteq O^{n+1}$ for every $n \in \mathbb{N}$.

Let us define the sequence $\{n_i\}_i \subseteq \mathbb{N}$ inductively as follows.

Let $n_1 = 1$. Let us assume that $n_{i-1}$ is defined for $i \geq 2$. Since $C$ is compact, $C^{n_{i-1}} \cup C^i$ is compact, so $C^{n_{i-1}} \cup C^i \subseteq \bigcup_{n \in \mathbb{N}} O^n$ yields the existence of finitely many natural numbers $i_1, \ldots, i_s$ such that $C^{n_{i-1}} \cup C^i \subseteq O^{i_1} \cup \ldots \cup O^{i_s}$. As $O^n \subseteq O^{n+1}$ for every $n \in \mathbb{N}$, then $C^{n_{i-1}} \cup C^i \subseteq O^{n_i}$ where $n_i = \max \{i_1, \ldots, i_s, n_{i-1}\}$.

Finally, by the definition of the $n_i$’s and $O \subseteq C$, it follows directly that $C^{n_i} \subseteq C^{n_{i+1}}$ for every $i \in \mathbb{N}$, and $L = \bigcup_{i \in \mathbb{N}} C^{n_i}$.

**Proposition 5.3.7.** Let $U$ be a connected abelian $\bigvee$-definable group such that $U^{00}$ exists. Let $X \subseteq U$ be a definable set such that $U^{00} \subseteq X$ and $Z \subseteq U$ a definable set. Then
(i) \( \mathcal{U} = \bigcup_{n \in \mathbb{N}} X^n \).

(ii) There is \( k \in \mathbb{N} \) such that \( Z \subseteq X^k \).

(iii) There is \( k \in \mathbb{N} \) such that the convex hull \( \text{ch}(Z) \) of \( Z \) is contained in \( X^k \). If, moreover, \( \mathcal{U} \) is torsion free, then \( \text{ch}\left(\frac{Z}{k}\right) \subseteq X \).

Proof. Let \( L \) denote the group \( \mathcal{U}/\mathcal{U}^0 \), let \( \pi : \mathcal{U} \rightarrow L \) be the quotient homomorphism, and consider \( L \) as the locally compact topological space given by the logic topology (see Fact 2.2.5). By [17, Thm. 3.9], \( L \) is isomorphic, as a topological group, to \( \mathbb{R}^{r_1} \times \mathbb{T}^{r_2} \) for some \( r_1, r_2 \in \mathbb{N} \).

By Remark 5.3.4 and saturation, there is a definable \( Y \subseteq X \) such that \( \mathcal{U}^0 \subseteq Y \subseteq Y \cdot Y \subseteq X \). Thus, by [17, Fact 2.3(2)], \( Y \) generates \( \mathcal{U} \). Furthermore, \( \pi'(Y) = \{ l \in L : \pi^{-1}(l) \subseteq Y \} \) is an open neighbourhood of the identity element \( e_L \) of \( L \). Therefore, \( \pi(Y) \) is a compact connected neighbourhood of \( e_L \) in \( L \) and generates \( L \).

Claim 5.3.6 yields the existence of an increasing sequence \( \{ n_i \}_{i \in \mathbb{N}} \subseteq \mathbb{N} \) such that \( \pi(Y)^{n_i} \subseteq \pi(Y)^{n_{i+1}} \) for every \( i \in \mathbb{N} \) and \( L = \bigcup_{i \in \mathbb{N}} \pi(Y)^{n_i} = \pi\left( \bigcup_{i \in \mathbb{N}} Y^{n_i} \right) \). Hence, \( \mathcal{U} = \bigcup_{i \in \mathbb{N}} Y^{n_i} \cdot \mathcal{U}^0 = \bigcup_{i \in \mathbb{N}} (Y \cdot \mathcal{U}^0)^{n_i} \subseteq \bigcup_{i \in \mathbb{N}} X^{n_i} \subseteq \bigcup_{n \in \mathbb{N}} X^n \). This gives us (i).

Since \( \pi(X) \) is a compact set in \( L \) and \( L = \bigcup_{i \in \mathbb{N}} \pi(Y)^{n_i} \), there are \( k_1, \ldots, k_s \in \{ n_i \}_{i \in \mathbb{N}} \) such that \( \pi(X) \subseteq \pi(Y)^{k_1} \cup \ldots \cup \pi(Y)^{k_s} \). As \( \pi(Y)^{n_i} \subseteq \pi(Y)^{n_{i+1}} \), then \( \pi(X) \subseteq \pi(Y)^{k^*} \) where \( k^* = \max\{ k_1, \ldots, k_s \} \). Therefore,

\[
X \subseteq X \cdot \mathcal{U}^0 \subseteq Y^{k^*} \cdot \mathcal{U}^0 \subseteq (Y \cdot Y)^{k^*} \subseteq X^{k^*}.
\]

By (i) and saturation, if \( Z \subseteq \mathcal{U} \) is a definable set, then there are \( l_1, \ldots, l_m \in \mathbb{N} \) such that \( Z \subseteq X^{l_1} \cup \ldots \cup X^{l_m} \), then \( Z \subseteq X^{k^{**}} \) where \( k^{**} = \max\{ l_1 k^*, \ldots, l_m k^* \} \), which yields (ii).

Finally, let us prove (iii). By Lemma 3.7 in [17], \( \mathcal{U}^0 \) exists if and only if \( \mathcal{U} \) covers a definable group, and by Theorem 5.6 in [4], if and only if \( \mathcal{U} \) has definably bounded convex hulls; i.e., for every definable \( Z' \subseteq \mathcal{U} \) there is a definable \( W \subseteq \mathcal{U} \) containing the convex hull \( \text{ch}(Z') \) of \( Z' \). Then, there is a definable set \( W \) such that \( \text{ch}(Z) \subseteq W \subseteq \mathcal{U} \), and (ii) yields the existence of \( k \in \mathbb{N} \) such that \( W \subseteq X^k \), then \( \text{ch}(Z) \subseteq X^k \). Since \( \mathcal{U}^0 \) exists, Proposition 3.5 in [17] implies that \( \mathcal{U} \) is divisible. If in addition \( \mathcal{U} \) is torsion free, then the map \( x \mapsto x^k : \mathcal{U} \rightarrow \mathcal{U} \) is a group isomorphism for every \( k \in \mathbb{N} \), so if \( \text{ch}(Z) \subseteq X^k \), then \( \text{ch}\left(\frac{Z}{k}\right) \subseteq X \). \( \square \)
5.4 Local homomorphisms and generic sets: some technical propositions

Below we prove some technical results that will be applied in the proofs of Theorems 5.5.1, 5.6.1, and 6.0.3. The first of these deals with the image of generic and type-definable sets by a local homomorphism, and the second one establishes a sufficient condition for which a locally definable covering homomorphism has a restriction that is both a definable homeomorphism and a local homomorphism in both directions.

**Proposition 5.4.1.** Let $\mathcal{Z}$ and $\mathcal{V}$ be locally definable groups such that $\mathcal{Z}^{00}$ exists. Let $W \subseteq \mathcal{Z}$ be a definable set such that $\mathcal{Z}^{00} \subseteq W$, and $\theta : W \to \mathcal{V}$ be a definable local homomorphism. Then

(i) there is a definable symmetric set $W' \subseteq W$ such that $\mathcal{Z}^{00} \subseteq W' \subseteq \prod_4 W' \subseteq W$ and $\theta (W')$ is generic in $\langle \theta (W') \rangle$.

(ii) $\theta (\mathcal{Z}^{00})$ is a type-definable subgroup of $\langle \theta (W') \rangle$ of index less than $\kappa$, and hence $\langle \theta (W') \rangle^{00} \subseteq \theta (\mathcal{Z}^{00}) \subseteq \theta (W')$.

**Proof.** (i) By Remark 5.3.5 and saturation, there is a definable symmetric $W' \subseteq W$ such that $\mathcal{Z}^{00} \subseteq W' \subseteq \prod_4 W' \subseteq W$. Since $W'$ is generic in $\mathcal{Z}$ and the structure is $\kappa$-saturated (with $\kappa \geq \aleph_1$), then $W'W' \subseteq \bigcup_{i<\kappa} w_i W$ for some $\{w_i\}_{i<\kappa} \subseteq \mathcal{Z}$.

Let $I = \{i < \aleph_1 : W'W' \cap w_i W' \neq \emptyset\}$, and $i \in I$. If $xy = w_iz$ with $x, y, z \in W'$, then $w_i = x y z^{-1} \in \prod_3 W'$, thus $w_i W' \subseteq \prod_4 W' \subseteq W$. Therefore,

$$\theta (W'W') = \theta (W') \theta (W') \subseteq \bigcup_{i \in I} \theta (w_i) \theta (W') \subseteq \langle \theta (W') \rangle,$$

and $\theta (w_i) \in \langle \theta (W') \rangle$ for $i \in I$. Hence, by Remark 3.2.4, $\theta (W')$ is a definable generic subset in $\langle \theta (W') \rangle$.

(ii) We will see that $\theta (\mathcal{Z}^{00})$ is a type-definable subgroup of $\langle \theta (W') \rangle$ of index less than $\kappa$. By saturation, $\theta (\mathcal{Z}^{00})$ is a type-definable set. Now, as $[\mathcal{Z} : \mathcal{Z}^{00}] < \kappa$, $W' \subseteq \bigcup_{j < \kappa} b_j \mathcal{Z}^{00}$ with $\{b_j\}_{j < \kappa} \subseteq \mathcal{Z}$. Let $J = \{j < \kappa : W' \cap b_j \mathcal{Z}^{00} \neq \emptyset\}$. Then, if $j \in J$ and $x = b_j z$ with $x \in W'$, $z \in \mathcal{Z}^{00}$, then $b_j = x z^{-1} \in W' \mathcal{Z}^{00} \subseteq \prod_2 W'$, so
\(\theta(b_j) \in \prod_2 \theta(W').\) Thus,

\[
\theta(W') \subseteq \bigcup_{j \in J} \theta(b_j) \theta(Z^{00}), \quad \text{and} \quad \theta(b_j) \in \langle \theta(W') \rangle.
\]

In addition, by (i), \(\langle \theta(W') \rangle \subseteq \bigcup_{i < \kappa} v_i \theta(W')\) for some \(\{v_i\}_{i < \kappa} \subseteq \langle \theta(W') \rangle\). Then,

\[
\langle \theta(W') \rangle \subseteq \bigcup_{i < \kappa} v_i \bigcup_{j \in J} \theta(b_j) \theta(Z^{00}) = \bigcup_{i < \kappa, j \in J} v_i \theta(b_j) \theta(Z^{00}).
\]

Hence, \(\langle \theta(W') \rangle : \theta(Z^{00}) \rangle < \kappa\).

Note that since \(\theta(Z^{00})\) is a type-definable subgroup of \(\langle \theta(W') \rangle\) of index \(< \kappa\), then \(\langle \theta(W') \rangle^{00}\) exists (see [19, Prop. 7.4]), and thus \(\langle \theta(W') \rangle^{00} \subseteq \theta(Z^{00}) \subseteq \theta(W')\). \(\square\)

**Proposition 5.4.2.** Let \(U, V\) be locally definable groups with identities \(e_U\) and \(e_V\), respectively, and \(\theta : U \to V\) a locally definable covering homomorphism. Let \(Y \subseteq V\) be a definable simply connected set, and \(Y' \subseteq Y\) a connected definable set such that \(e_Y \in Y'\) and \(Y'Y' \subseteq Y\), then there is a definable set \(W' \subseteq U\) such that \(e_U \in W'\), \(\theta |_{W'} : W' \to Y'\) is a definable homeomorphism and a local homomorphism in both directions.

![Diagram](attachment:image_url)

**Proof.** By Proposition 5.1.14, there is a definable \(W_1 \subseteq U\) open in \(\theta^{-1}(Y)\) such that \(e_U \in W_1\) and \(\theta |_{W_1^0} : W_1^0 \to Y\) is a definable homeomorphism, where \(W_1^0\) is the identity component of \(W_1\). Let \(W' = \theta |_{W_1^{-1}}(Y')\), then \(\theta(W'W') = \theta(W') \theta(W') \subseteq Y'Y' \subseteq Y\), then \(W'W' \subseteq \theta^{-1}(Y) \subseteq W_1^0 \ker(\theta)\).

In addition, \(W_1^0 k_1 \cap W_1^0 k_2 = \emptyset\) if \(k_1 \neq k_2\) and \(k_1, k_2 \in \ker(\theta)\); otherwise, if there are \(y_1, y_2 \in W_1^0\) such that \(y_1 k_1 = y_2 k_2\), then \(\theta(y_1 k_1) = \theta(y_1) = \theta(y_2 k_2) = \theta(y_2)\), but \(\theta\) is injective in \(W_1^0\), then \(y_1 = y_2\), so \(k_1 = k_2\), which is a contradiction since \(k_1 \neq k_2\). Then \(\theta |_{W_2} : W_2 \to Y\) is a definable covering map of ld-spaces and \(e_U \in W_2\).
Therefore, from the connectedness of $W'W'$ and $W'W' \subseteq W_1^0 \ker(\theta)$, we get $W'W' \subseteq W_1^0$. Thus, Remark 4.2.4 implies that the homeomorphism $\theta |_{W'}: W' \rightarrow Y'$ is a local homomorphism in both directions. \hfill \Box

### 5.5 Extension of a definable local homomorphism from a torsion free abelian locally definable group

**Theorem 5.5.1.** Let $Z$ be a connected abelian torsion free locally definable group such that $Z^{00}$ exists, and let $V$ be an abelian locally definable group. Let $W \subseteq Z$ be a definable set such that $Z^{00} \subseteq W$. Assume that $\theta : W \subseteq Z \rightarrow V$ is a definable local homomorphism.

Then there exists a unique locally definable homomorphism $\bar{\theta} : Z \rightarrow V$ extending $\theta$.

If in addition $V$ is connected, $V^{00}$ exists, $V^{00} \subseteq \theta(W)$, $\theta$ is injective and $\theta^{-1} : \theta(W) \rightarrow W$ is a local homomorphism, then $\bar{\theta} : Z \rightarrow \langle \theta(W) \rangle = V$ is the o-minimal universal covering homomorphism of $V$.

\[
\begin{array}{c}
\xrightarrow{\bar{\theta}} \quad \xleftarrow{\theta} \\
Z \quad \bigcup \quad V \\
\end{array}
\]

\begin{proof}
By Proposition 5.4.1, there is a definable symmetric $W_1 \subseteq W$ such that $Z^{00} \subseteq W_1 \subseteq \prod_i W_1 \subseteq W$, and $\theta(W_1)$ is generic in $\langle \theta(W) \rangle$. Now, note that since $Z^{00}$ exists, by [17, Proposition 3.5], $Z$ is divisible, then the map $z \mapsto z^k : Z \rightarrow Z$ is a group isomorphism. By Proposition 5.3.7(iii), there is $k \in \mathbb{N}$ such that the convex hull $\text{ch} \left( W_1^{\frac{1}{k}} \right)$ of $W_1^{\frac{1}{k}}$ is contained in $W_1$.

Let $y \in Z = \langle W_1 \rangle = \bigcup_{n \in \mathbb{N}} \prod_n W_1$, so there are $n \in \mathbb{N}$ and $y_1, \ldots, y_n \in W_1$ such that $y = y_1 \cdots y_n$. Let

$$\bar{\theta}(y) = \left( \theta \left( y_1^{\frac{1}{k}} \right) \cdots \theta \left( y_n^{\frac{1}{k}} \right) \right)^k.$$
Claim 5.5.2. The map \( \overline{\theta} : \mathcal{Z} \to \mathcal{V} \) defined as above satisfies the following.

(i) \( \overline{\theta} \) is a well defined map.

(ii) \( \overline{\theta} \) is a locally definable homomorphism.

(iii) \( \overline{\theta} \) is the unique extension of \( \theta : W \to \mathcal{V} \) that is a locally definable homomorphism from \( \mathcal{Z} \) to \( \mathcal{V} \).

(iv) If, moreover, \( \mathcal{V} \) is connected, \( \mathcal{V}^{00} \) exists, \( \mathcal{V}^{00} \subseteq \theta(W) \), \( \theta \) is injective and \( \theta^{-1} : \theta(W) \to W \) is a local homomorphism, then \( \overline{\theta}(\mathcal{Z}) = \mathcal{V} \) and \( \overline{\theta} \) is the o-minimal universal covering homomorphism of \( \mathcal{V} \).

Proof. (i) As \( \operatorname{ch}(W_{1}^{\frac{1}{i}}) \subseteq W_{1} \), then for every \( i,j \in \mathbb{N} \) with \( j \leq i \) and \( i > 0 \) we have that:

\[
\prod_{j} \left( W_{1}^{\frac{1}{i}} \right)^{\frac{1}{j}} \subseteq W_{1}.
\]

And since \( \theta \) is a locally homomorphism, then for every \( y_{1}, \ldots, y_{j} \in W_{1}^{\frac{1}{i}} \)

\[
\theta \left( y_{1}^{\frac{1}{i}} \cdots y_{j}^{\frac{1}{i}} \right) = \theta \left( y_{1}^{\frac{1}{i}} \right) \cdots \theta \left( y_{j}^{\frac{1}{i}} \right). \tag{5.5.0.1}
\]

Now, we will see that \( \overline{\theta} \) is well defined.

Let \( y \in \mathcal{Z} = \langle W_{1} \rangle = \bigcup_{n \in \mathbb{N}} \prod_{n} W_{1} \), and suppose that

\[
y = y_{1} \cdots y_{n} = x_{1} \cdots x_{m} \tag{5.5.0.2}
\]

for some \( y_{1}, \ldots, y_{n}, x_{1}, \ldots, x_{m} \in W_{1} \), and \( m, n \in \mathbb{N} \). Additionally, assume, without
loss of generality, that $m \leq n$.

\[
\overline{\theta} (y_1 \cdots y_n) = \left( \theta \left( y_1^{\frac{1}{k_1}} \right) \cdots \theta \left( y_n^{\frac{1}{k_n}} \right) \right)^k \\
= \left( \theta \left( y_1^{\frac{1}{k_1n}} \cdots y_n^{\frac{1}{k_nn}} \right) \right) \cdots \theta \left( y_1^{\frac{1}{k_1n}} \cdots y_n^{\frac{1}{k_nn}} \right)^k, \text{ by Eq. (5.5.0.1)} \\
= \left( \left( \theta \left( y_1^{\frac{1}{k_1n}} \right) \right)^n \cdots \theta \left( y_n^{\frac{1}{k_nn}} \right)^n \right)^k, \text{ by Eq. (5.5.0.1)} \\
= \left( \left( \theta \left( x_1^{\frac{1}{k_1n}} \cdots x_m^{\frac{1}{k_mn}} \right) \right)^n \right)^k, \text{ by Eq. (5.5.0.1)} \\
= \left( \left( \theta \left( x_1^{\frac{1}{k_1n}} \right) \right)^n \cdots \theta \left( x_m^{\frac{1}{k_mn}} \right)^n \right)^k, \text{ by Eq. (5.5.0.1)} \\
= \left( \theta \left( x_1^{\frac{1}{k}} \right) \cdots \theta \left( x_m^{\frac{1}{k}} \right) \right)^k \\
= \overline{\theta} (x_1 \cdots x_m).
\]

Therefore, $\overline{\theta}$ is well defined.

(ii) Since $\overline{\theta} \mid_{\prod_n W_i}$ is a definable map for every $n \in \mathbb{N}$, then the restriction of $\overline{\theta}$ to a definable subset of $Z = \langle W_1 \rangle$ is a definable map. And by definition of $\overline{\theta}$, $\overline{\theta}$ is clearly a group homomorphism.

(iii) First, we will see that $\theta \mid_{ch(\overline{W_1^k})} = \overline{\theta} \mid_{ch(\overline{W_1^k})}$.

By definition of the convex hull (Def. 5.3.2) and the divisibility of $Z$, an element in $ch\left( W_1^{\frac{1}{k}} \right) \subseteq W_1$ is of the form $y_1^{\frac{1}{n}} \cdots y_n^{\frac{1}{n}}$ for some $y_1, \ldots, y_n \in W_1^{\frac{1}{k}}$ and $n \in \mathbb{N}$. Thus,
\[ \vartheta \left( y_1^{\frac{1}{n}} \cdots y_n^{\frac{1}{n}} \right) = \left( \theta \left( y_1^{\frac{1}{nk}} \cdots y_n^{\frac{1}{nk}} \right) \right)^k, \text{ by Eq. (5.5.0.1)} \]
\[ = \left( \theta \left( y_1^{\frac{1}{nk}} \right) \cdots \theta \left( y_n^{\frac{1}{nk}} \right) \right)^k \]
\[ = \left( \theta \left( y_1^{\frac{1}{nk}} \right) \right)^k \cdots \left( \theta \left( y_n^{\frac{1}{nk}} \right) \right)^k, \text{ by Eq. (5.5.0.1)} \]
\[ = \theta \left( y_1^{\frac{1}{n}} \right) \cdots \theta \left( y_n^{\frac{1}{n}} \right), \text{ by Eq. (5.5.0.1)} \]
\[ = \theta \left( y_1^{\frac{1}{n}} \cdots y_n^{\frac{1}{n}} \right). \]

Then \( \theta \) and \( \vartheta \) agree on \( ch \left( W_1^{\frac{1}{n}} \right) \).

Now, we will verify uniqueness. Let \( \beta : Z \to V \) be a locally definable homomorphism that is an extension of \( \theta : W \to V \), then in particular \( \beta \) and \( \bar{\vartheta} \) agree on \( W_1^{\frac{1}{n}} \). Let \( Z' = \{ z \in Z : \beta (z) = \bar{\vartheta} (z) \} \). Then \( G^{00} \subseteq W_1^{\frac{1}{n}} \subseteq Z' \) and \( Z' \) is an open locally definable subgroup of \( Z \). By [11, Lemma 2.12], \( Z' \) is a compatible subset of \( Z \). But \( Z \) is connected, then \( Z = Z' \); i.e., \( \beta = \bar{\vartheta} \).

Then, \( \beta = \bar{\vartheta} \). And hence, \( \bar{\vartheta} \mid_W = \theta \); i.e., \( \bar{\vartheta} \) is also an extension of \( \theta : W \to V \).

(iv) First, note that \( \bar{\vartheta} (Z) = \bigcup_{n \in \mathbb{N}} \prod_n \left( \theta \left( W_1^{\frac{1}{n}} \right) \right)^k = \langle \left( \theta \left( W_1^{\frac{1}{n}} \right) \right)^k \rangle \). Now, by Proposition 5.3.7(ii), there is \( l \in \mathbb{N} \) such that \( W \subseteq W_1^l \), then \( \theta^{-1} (V^{00}) \subseteq W \subseteq W_1^l \). By the hypothesis on \( \theta^{-1} \), \( \theta^{-1} (V^{00}) \) is a type-definable subgroup of \( Z \); moreover, by [17, Proposition 3.5], \( V^{00} \) is divisible, so \( \theta^{-1} (V^{00}) \) is an abelian torsion free divisible subgroup of \( Z \). Henceforth, \( \theta^{-1} (V^{00}) \subseteq W_1^l \) implies that \( \theta^{-1} (V^{00}) = \theta^{-1} (V^{00}) W_1^\frac{1}{l} \subseteq W_1^{\frac{1}{n}}, \) thus \( V^{00} \subseteq \left( \theta \left( W_1^{\frac{1}{n}} \right) \right)^k \), it follows that \( V = \langle \left( \theta \left( W_1^{\frac{1}{n}} \right) \right)^k \rangle = \langle \theta (W) \rangle \). Then, \( \bar{\vartheta} : Z \to V \) is surjective.

On the other hand, notice that \( \dim (\ker (\bar{\vartheta})) = 0 \) if and only if \( \dim (Z) = \dim (\bar{\vartheta} (Z)) = \dim (\langle \theta (W_1) \rangle) \). Since \( W_1 \) and \( \langle \theta (W_1) \rangle \) are generic in \( Z \) and \( \langle \theta (W_1) \rangle \), respectively, then \( \dim (Z) = \dim (W_1) \) and \( \dim (\langle \theta (W_1) \rangle) = \dim (\langle \theta (W_1) \rangle) \); finally, \( \dim (Z) = \dim (\langle \theta (W_1) \rangle) \) follows from the injectivity of \( \theta \), so \( \bar{\vartheta} : Z \to V \) is a locally definable covering homomorphism by [11, Theorem 3.6].

82
Since \( Z \) is abelian, connected, and \( Z^{00} \) exists, then \( Z \) covers an abelian definable group ([17, Thm. 3.9]). Thus Claim 5.2.8 yields \( Z \) is simply connected. Therefore, by [15, Remark 3.8], \( \theta : Z \to V \) is the o-minimal universal covering homomorphism of \( V \).

This concludes the proof of Theorem 5.5.1.

5.6 Universal covers of locally homomorphic abelian locally definable groups

**Theorem 5.6.1.** Let \( U, V \) be abelian connected locally definable groups such that \( U^{00} \) exists and \( U^{00} \) is an intersection of \( \omega \)-many simply connected definable subsets of \( U \). Let \( X \subseteq U \) be a definable set with \( U^{00} \subseteq X \), and \( \phi : X \subseteq U \to \phi(X) \subseteq V \) a definable homeomorphism and a local homomorphism. Assume that \( \pi : \tilde{V} \to V \) is the o-minimal universal covering homomorphism of \( V \).

Then,

(i) there are a connected locally definable subgroup \( W \) of \( \tilde{V} \) and a locally definable homomorphism \( \tilde{\theta} : W \to U \) that is the o-minimal universal covering homomorphism of \( U \), and

(ii) there is a connected symmetric definable \( X' \subseteq X \) with \( U^{00} \subseteq X' \) such that \( \pi \mid_{W} : W \to \langle \phi(X') \rangle \leq V \) is the o-minimal universal covering homomorphism of \( \langle \phi(X') \rangle \).

If in addition \( X \) is simply connected, then \( W \) is a subgroup of the o-minimal universal covering group \( \langle \phi(X) \rangle \) of \( \langle \phi(X) \rangle \).
Proof. Since $U^{00}$ is an intersection of $\omega$-many simply connected definable subsets of $U$, then saturation and Remark 5.3.5 imply that there are simply connected definable sets $X_1$ and $X_2$ such that $U^{00} \subseteq X_2 \subseteq X_2X_2 \subseteq X_1 \subseteq X$. Thus, by Remark 4.2.4, $\phi \mid_{X_2}$ is a local homomorphism in both directions.

Moreover, the connected definable set $Y_2 = \phi (X_2)$ is such that $Y_2Y_2 \subseteq Y_1$ where $Y_1 = \phi (X_1)$ and the identity of $V$ belongs to $Y_2$. So Proposition 5.4.2 yields the existence of a connected definable $W_2 \subseteq \tilde{V}$ such that the identity of $\tilde{V}$ is in $W_2$ and $\pi \mid_{W_2} : W_2 \rightarrow Y_2$ is a definable homeomorphism and a local homomorphism in both directions, hence so is $\theta = \phi \mid_{X_2}^{-1} \circ \pi \mid_{W_2}$.

By Remark 5.3.5 and Proposition 5.4.1, there is a connected symmetric definable $X' \subseteq X_2$ such that (i) $U^{00} \subseteq X' \subseteq \prod X' \subseteq X_2$, (ii) $W_3 = \theta^{-1} (X')$ is generic in $\mathcal{W} = \langle W_3 \rangle \leq \tilde{V}$, $\mathcal{W}^{00}$ exists, and $\mathcal{W}^{00} \subseteq W_3$, and (iii) $Y_3 = \phi (X')$ is generic in $\langle Y_3 \rangle \leq V$, $\langle Y_3 \rangle^{00}$ exists, and $\langle Y_3 \rangle^{00} \subseteq Y_3$. Note that $\mathcal{W}$ is connected by Proposition 5.1.8.

By Theorem 5.5.1, there are locally definable homomorphisms $\bar{\theta} : \mathcal{W} \rightarrow U$ and $\bar{\pi} : \mathcal{W} \rightarrow \langle Y_3 \rangle \leq V$ that are the o-minimal universal covering homomorphisms of $U$ and $\langle Y_3 \rangle$, respectively. Moreover, $\bar{\theta} : \mathcal{W} \rightarrow U$ and $\bar{\pi} : \mathcal{W} \rightarrow \langle Y_3 \rangle \leq V$ are the unique extensions of $\theta \mid_{W_3} : W_3 \rightarrow U$ and $\pi \mid_{W_3} : W_3 \rightarrow Y_3 \subseteq V$, respectively, that are locally definable homomorphisms. Since $\pi \mid_{W_3} = \pi \mid_{W_3}$, then $\bar{\pi} = \bar{\pi} \mid_{\mathcal{W}}$. If in addition $X$ is simply connected, then the o-minimal universal covering group $\langle \phi (X) \rangle$ of $\langle \phi (X) \rangle$ exists, and $\mathcal{W}$ is a closed subgroup of $\langle \phi (X) \rangle$.

\[ \square \]

5.6.1 The o-minimal universal covering group of an abelian definably connected definably compact semialgebraic group

Applying the main results obtained so far, we present below one of the key results of this work.

**Theorem 5.6.2.** Let $G$ be an abelian definably connected definably compact group definable in a sufficiently saturated real closed field $R$. Then there are a Zariski-connected $R$-algebraic group $H$, an open connected locally definable subgroup $\mathcal{W}$ of the o-minimal universal covering group $\widetilde{H (R)}$ of $H (R)^0$, and a locally definable
homomorphism \( \overline{\vartheta} : \mathcal{W} \subseteq \tilde{H}(R)^0 \rightarrow G \) that is the o-minimal universal covering homomorphism of \( G \).

**Proof.** By Theorem 4.3.1, there are a Zariski-connected \( R \)-algebraic group \( H \) such that \( \dim(G) = \dim(H(R)) \), a definable set \( X \subseteq G \) such that \( G^{00} \subseteq X \), and a definable homeomorphism \( \phi : X \subseteq G \rightarrow \phi(X) \subseteq H(R) \) that is also a local homomorphism.

By [3, Thm. 2.2], \( G^{00} \) is an intersection of \( \omega \)-many simply connected definable subsets of \( G \). Thus, by Theorem 5.6.1, there are a connected locally definable subgroup \( \mathcal{W} \leq \tilde{H}(R)^0 \) and a locally definable homomorphism \( \overline{\vartheta} : \mathcal{W} \subseteq \tilde{H}(R)^0 \rightarrow G \) that is the o-minimal universal covering homomorphism of \( G \), and since \( \dim(G) = \dim(\mathcal{W}) \), \( \mathcal{W} \) is also open in \( \tilde{H}(R)^0 \). \( \square \)
Chapter 6

Characterization of the 1-dimensional semialgebraic groups

In this chapter we offer a characterization of the o-minimal universal covering group of a one-dimensional definably connected group definable in a sufficiently saturated real closed field. This result can be seen as a version of Theorem 5.6.2 when the definable group $G$ has dimension one.

**Theorem 6.0.3.** Let $G$ be a one-dimensional definably connected group definable in a sufficiently saturated real closed field $\mathcal{R} = (\mathbb{R}, <, +, 0, \cdot, 1)$. Then the o-minimal universal covering group $\tilde{G}$ of $G$ is locally $\mathcal{R}$-definably isomorphic, as a locally definable group, to a connected locally definable group $\mathcal{W}$ that is either the o-minimal universal covering group $\tilde{H}(\mathbb{R})^0$ of $H(\mathbb{R})^0$ or an open subgroup of $H(\mathbb{R})^{00}$ for some one-dimensional Zariski-connected $\mathbb{R}$-algebraic group $H$.

**Proof.** If $G$ is torsion free, $G$ is already its own o-minimal universal covering group because of Claim 5.2.8, and by Facts 2.3.1 and 2.3.2, $G$ is $\mathcal{R}$-definably isomorphic to $(\mathbb{R}, +)$ or $(\mathbb{R}^{>0}, \cdot)$. Thus we can assume that $G$ is definably compact.

By Theorem 4.3.1 and Remark 4.3.8, there are a Zariski-connected $\mathbb{R}$-algebraic group $H$, a definably connected symmetric definable set $X \subseteq G$ with $G^{00} \subseteq X$, and a definable homeomorphism $\phi : X \subseteq G \rightarrow \phi(X) \subseteq H(\mathbb{R})^0$ that is a local homomorphism in both directions.
By Proposition 5.4.1, there is a definably connected symmetric definable set $X_1 \subseteq X$ such that $G^{00} \subseteq X_1$, $(\phi(X_1))^{00}$ exists, $\phi(G^{00})$ is a type-definable subgroup of $(\phi(X_1))$ of index less than $\kappa$, and $\phi : X_1 \rightarrow \phi(X_1)$ is a local homomorphism in both directions.

**Claim 6.0.4.**

(i) If $H(R)^0$ is torsion free, then there is a locally definable map $\bar{\phi}^{-1} : (\phi(X_1)) \subseteq H(R)^0 \rightarrow G$ that is the o-minimal universal covering homomorphism of $G$.

(ii) If $H(R)^0$ is definably compact, then either there is a locally definable map $\bar{\phi}^{-1} : (\phi(X_1)) \subseteq H(R)^{00} \rightarrow G$ that is the o-minimal universal covering homomorphism of $G$, or there is a locally definable map $\bar{\phi}^{-1} : H(R)^0 \rightarrow G$ that is the o-minimal universal covering homomorphism of $G$.

**Proof.** (i) Since $(\phi(X_1)) \subseteq H(R)^0$ is also torsion free, [17, Proposition 3.6] yields $(\phi(X_1))^{00} = \phi(G^{00})$. Proposition 5.3.7 implies that there is $k \in \mathbb{N}$ such that $\text{ch}(\phi(X_1)^{1/k}) \subseteq \phi(X_1)$. This let us to prove that the map $\bar{\phi}^{-1} : (\phi(X_1)) \subseteq H(R)^0 \rightarrow G$ given by

$$
\bar{\phi}^{-1}(y) = \left(\phi^{-1}\left(y_1^{1/k}\right) \cdots \phi^{-1}\left(y_n^{1/k}\right)\right)^k
$$

for $y = y_1 \cdots y_n$ and $y_1, \ldots, y_n \in \phi(X_1)$ is a well defined map exactly as in Claim 5.5.2, and with the same proof of 5.5.2, we have that $\bar{\phi}^{-1} : (\phi(X_1)) \subseteq H(R)^0 \rightarrow G$ is the o-minimal universal covering homomorphism of $G$.

(ii) We can assume that the generic set $X_1$ has no nontrivial element of order two (otherwise, if $a_0$ is the nontrivial element of order two of $G$, by Claim 4.3.7, there is a definably connected symmetric definable set $X_2 \subseteq X_1 \setminus \{a_0\}$ with $G^{00} \subseteq X_2$ and we replace $X_1$ by $X_2$), so does $\phi(X_1) \subseteq H(R)^0$. By Proposition 2.3.4, then either $\phi(X_1) \subseteq H(R)^{00}$ or $H(R)^{00} \subseteq \phi(X_1)$.

Let us suppose first that $\phi(X_1) \subseteq H(R)^{00}$, since $H(R)^{00}$ is torsion free, exactly as in the case of $H(R)^0$ torsion free in (i), we conclude that there is a locally definable map $\bar{\phi}^{-1} : (\phi(X_1)) \subseteq H(R)^{00} \rightarrow G$ that is the o-minimal universal covering homomorphism of $G$.

Finally, if $H(R)^{00} \subseteq \phi(X_1)$, then $(\phi(X_1)) = H(R)^0$. Let $\pi : \overset{\sim}{H(R)^0} \rightarrow H(R)^0$.
be the o-minimal universal covering homomorphism of $H(R)^0$. In this case, we will see that $\tilde{H}(R)^0$ will be also an o-minimal universal covering group of $G$.

Since $G^{00}$ is an intersection of $\omega$-many simply connected definable subsets of $G$ ([3, Thm. 2.2]), saturation and Remark 5.3.5 imply that there are simply connected definable sets $X_2$ and $X_3$ such that $G^{00} \subseteq X_3 \subseteq X_3 X_2 \subseteq X_2 \subseteq X_1$. Moreover, the connected definable set $Y_3 = \phi(X_3)$ is such that $Y_3 X_3 \subseteq Y_2$ where $Y_2 = \phi(X_2)$ and the identity of $H(R)^0$ belongs to $Y_3$. Then, by Proposition 5.4.2, there is a connected definable $W_3 \subseteq \tilde{H}(R)^0$ such that the identity of $\tilde{H}(R)^0$ is in $W_3$ and $\pi_{|W_3}: W_3 \to Y_3$ is a definable homeomorphism and a local homomorphism in both directions, hence so is $\theta = \phi_{|X_3}^{-1} \circ \pi_{|W_3}$. By Remark 5.3.5 and Proposition 5.4.1, there is a connected symmetric definable $X_4 \subseteq X_3$ such that (i) $G^{00} \subseteq X_4 \subseteq \prod X_4 \subseteq X_3$, (ii) $W_4 = \theta^{-1}(X_4)$ is generic in $W = \langle W_4 \rangle \leq H(R)^0$, $W^{00}$ exists, and $W^{00} \subseteq W_4$, and (iii) $Y_4 = \phi(X_4) \supseteq \phi(G^{00})$, as $H(R)^{00} = \langle \phi(X_1) \rangle \subseteq \phi(G^{00})$, then $H(R)^{00} \subseteq Y_4$. Note that $W$ is connected by Proposition 5.1.8. By Theorem 5.5.1, there are locally definable homomorphisms $\overline{\theta}: W \to G$ and $\pi_{|W}: W \to \langle Y_4 \rangle = H(R)^0$ that are the o-minimal universal covering homomorphisms of $G$ and $H(R)^0$, respectively. Then, $W = H(R)^0$. Thus we finish the proof of Claim 6.0.4.

This ends the proof of Theorem 6.0.3.

We recall, by Fact 2.5.2, that if $R$ is a real closed field and $H$ is a 1-dimensional Zariski-connected $R$-algebraic group, then the group $H(R)^0$ is $R$-definably isomorphic, as a $R$-definable group, to $(R, +)$, $(R^{>0}, \cdot)$, $SO(2, R)$, or $E(R)^0$ for some elliptic curve $E$ defined over $R$.

**Definition 6.0.5.** Let $U$ be a $\bigvee$-definable group and $\Lambda \subseteq U$ a normal subgroup, we say that the quotient $U/\Lambda$ is definable if there exists a definable group $G$ and a surjective $\bigvee$-definable homomorphism $\mu: U \to G$ whose kernel is $\Lambda$.

An example of a definable quotient is the quotient of the o-minimal universal covering group $\tilde{G}$ of a definably connected definable group $G$ by the kernel of its o-minimal universal covering homomorphism.

88
Recall, by Fact 2.3.1, that any one-dimensional definably connected group $G$ definable in a real closed field $R = (R, <, +, 0, \cdot, 1)$ is either $R$-definably compact or torsion free. In the latter case, $G$ is $R$-definably isomorphic, as a definable group, to $(R, +)$ or $(R^{>0}, \cdot)$ by Facts 2.3.1 and 2.3.2. For the $R$-definably compact case we have the following.

**Remark 6.0.6.** Let $G$ be a one-dimensional definably compact definably connected group definable in a sufficiently saturated real closed field $R = (R, <, +, 0, \cdot, 1)$, then there are

(i) a one-dimensional Zariski-connected $R$-algebraic group $H$,

(ii) a locally $R$-definable ordering $<_\sim$ making $\left(\widetilde{H(R)}^0, <_\sim, \oplus\right)$ into an ordered group, where $\left(\widetilde{H(R)}^0, \oplus\right)$ is the o-minimal universal covering group of $H(R)^0$,

(iii) an element $a \in H(R)^0$ other than the identity element $e'$ of $H(R)^0$ with $e' <_\sim a$ such that

$G$ is $R$-definably isomorphic to the $R$-definable group $([e', a], \oplus_{mod\, a})$ with addition modulo $a$ defined as follows

$$x \oplus_{mod\, a} y = \begin{cases} x \oplus y & \text{if } x \oplus y <_\sim a \\ x \oplus y \ominus a & \text{if } a \leq_\sim x \oplus y \end{cases}.$$  

**Proof.** By Theorem 5.6.2, there are a connected one-dimensional $R$-algebraic group $H$, an open locally definable subgroup $\widetilde{G}$ of $H(R)^0$, and a locally definable homomorphism $\pi : \widetilde{G} \to G$ that is the o-minimal universal covering homomorphism of $G$. Thus, $G$ is $R$-definably isomorphic to the definable quotient group $\widetilde{G}/\ker(\pi)$.

As $\left(\widetilde{H(R)}^0, \oplus\right)$ is a locally definable torsion free group, Fact 2.3.1 yields the existence of a locally $R$-definable ordering $<_\sim$ making $\left(\widetilde{H(R)}^0, <_\sim, \oplus\right)$ into an ordered group; moreover, the order topology given by $<_\sim$ on $H(R)^0$ agrees with the
the \( \tau \)-topology ([32]) of \( \widetilde{H}(R)^0 \) as a locally definable group. Also, since the \( \tau \)-topology of \( \widetilde{G} \) as a locally definable group agrees with the subspace topology induced on \( \widetilde{G} \) by \( H(R)^0 \) ([32, Lemma 2.6]), then the order topology given by \( \preceq \) on \( \widetilde{G} \) agrees with the \( \tau \)-topology of \( \widetilde{G} \) as a locally definable group.

From the proof of Theorem 6.0.3, we know that \( \widetilde{G} \) is definably generated by a definably connected symmetric definable set, which we can assume without loss of generality, equal to \( \langle \oplus b, b \rangle \) for some \( b \in G \) and such that \( \langle \oplus b, b \rangle \cap \ker(\pi) = \{e'\} \), thus \( \widetilde{G} = \bigcup_{b \in \mathbb{N}} \langle \oplus nb, nb \rangle \). Since \( \bigcup_{b \in \mathbb{N}} \pi((\oplus nb, nb)) = G \), saturation yields the existence of some \( N \in \mathbb{N} \) such that \( \pi((\oplus Nb, Nb)) = G \). Then \( \langle \oplus Nb, Nb \rangle \oplus \ker(\pi) = \widetilde{G} \).

Let \( a = \min \{x \in \ker(\pi) : e' \preceq x\} \). Then \( b < a \) since \( \langle \oplus b, b \rangle \cap \ker(\pi) = \{e'\} \), therefore \( \langle \oplus Na, Na \rangle \oplus \ker(\pi) = \widetilde{G} \).

Now note that, by [14, Thm. 3.11] (see Fact 5.2.4), \( \ker(\pi) \cong \pi_1(G) \), and by [16, Thm. 1.1], the o-minimal fundamental group \( \pi_1(G) \) of \( G \) is isomorphic, as abstract group, to \( \mathbb{Z} \), so \( \mathbb{Z} \cong \ker(\pi) \leq H(R)^0 \). Therefore, \( \ker(\pi) = \langle a \rangle = \{la : l \in \mathbb{Z}\} \).

We will define a surjective locally \( \mathcal{R} \)-definable homomorphism \( \eta : (\widetilde{G}, \oplus) \rightarrow (\langle e', a \rangle, \oplus_{mod a}) \) with \( \ker(\eta) = \ker(\pi) \).

Note that for every element \( x \in (\oplus Na, Na) \setminus \langle a \rangle \) there is a unique \( m_x \in \mathbb{Z} \cap (-N, N) \) such that \( m_xa \preceq x <_\preceq (m_x + 1)a \).

Let \( \eta : (\widetilde{G}, \oplus) \rightarrow (\langle e', a \rangle, \oplus_{mod a}) \) be the map defined as \( \eta(g) = e' \) if \( g \in \ker(\pi) = \langle a \rangle \), and for \( g \in \widetilde{G} \setminus \ker(\pi) \), for which there is some \( x \in (\oplus Na, Na) \setminus \langle a \rangle \) such that \( g = x \oplus la \) for some \( l \in \mathbb{N} \) since \( (\oplus Na, Na) \oplus \ker(\pi) = \widetilde{G} \), define \( \eta(g) = \eta(x \oplus la) = x \oplus m_xa \) where \( m_x \) is the unique element in \( \mathbb{Z} \cap (-N, N) \) such that \( m_xa \preceq x <_\preceq (m_x + 1)a \).

**Claim 6.0.7.**

(i) \( \eta \) is well defined.

(ii) \( \eta \) is a surjective locally \( \mathcal{R} \)-definable map.

(iii) \( \eta \) is a group homomorphism.

**Proof.** (i) Let \( g \in \widetilde{G} \setminus \ker(\pi) \) such that \( g = x \oplus l_1a = y \oplus l_2a \). Since \( y = x \oplus (l_1 - l_2)a \) and \( m_ya \preceq y <_\preceq (m_y + 1)a \), then \( m_ya \preceq x \oplus (l_1 - l_2)a <_\preceq (m_y + 1)a \).
\[ x < (m_y - l_1 + l_2 + 1)a, \text{ then } m_x = m_y - l_1 + l_2. \] Thus \( \eta(x \oplus l_1a) = x \ominus m_xa = x \oplus (-m_y + l_1 - l_2)a = x \oplus (l_1 - l_2)a \ominus m_ya = y \ominus m_ya = \eta(y \oplus l_2a); \) i.e., \( \eta \) is well defined.

(ii) By definition of \( \eta \), \( \eta \) is a locally \( \mathcal{R} \)-definable map. As \( \eta \mid_{[e',a]} (g) = g \), \( \eta \) is surjective.

(iii) Let \( g_1 = x \oplus l_1a \), and \( g_2 = y \oplus l_2a \). Since \( m_xa < x < (m_x + l_1 + l_2 + 1)a \), then \( m_xa < y < (m_y + l_1 + l_2 + 1)a \). Thus we have two cases.

Case (i): \( (m_x + m_y)a < x \oplus y < (m_x + m_y + 1)a \Rightarrow e' < x \oplus y \ominus (m_x + m_y)a < a \). Then \( \eta(g_1 \oplus g_2) = \eta(x \oplus y \oplus (l_1 + l_2)a) = x \ominus m_x \oplus y \ominus m_ya = \eta(g_1) \oplus \mod_a \eta(g_2) \).

Case (ii): \( (m_x + m_y + 1)a < x \oplus y < (m_x + m_y + 2)a \Rightarrow 2a \). Then \( \eta(g_1 \oplus g_2) = \eta(x \oplus y \oplus (l_1 + l_2)a) = x \ominus m_xa \oplus y \ominus m_ya \ominus a = \eta(g_1) \oplus \mod_a \eta(g_2) \). Therefore, \( \eta \) is a group homomorphism.

\[ \eta : (\tilde{G}, \oplus) \to ([e',a], \oplus \mod_a) \] is a surjective locally \( \mathcal{R} \)-definable homomorphism with \( \ker(\eta) = \ker(\pi) \). Finally, as \( \mathcal{R} \) is \( \omega \)-minimal and \( \tilde{G} \) is a locally definable group, \( \tilde{G} \) has strong definable choice for definable families of subsets of \( \tilde{G} \) by [12, Corollary 8.1], it follows that \( G \) and \( ([e',a], \oplus \mod_a) \) are \( \mathcal{R} \)-definably isomorphic. This concludes the proof of this remark.
Appendix A

Proposition 3.1 of Hrushovski and Pillay in [21]

In this appendix we present to the reader the proof of Proposition 3.1 in [21] that was cited throughout the proof of Proposition 4.1.2. Proposition A.0.9 is a fundamental result to show Fact 1.0.1 ([21, Theorem A]).

We adapt the same notion of geometric substructure and structure defined by Hrushovski and Pillay in [21] (we recall it at the beginning of Sect. 4.1) as well as the same notation presented in Notation 4.1.1. We refer the reader to [21] for appropriate model-theoretic background.

We start off mentioning Fact A.0.8 that was proved by Hrushovski and is a model-theoretic version of the Theorem of Weil [49] about algebraic homogeneous spaces (see [8]). This fact will be applied in the first part of the proof of Proposition A.0.9. Essentially, we transcribed the proof of [21, Proposition 3.1], but sometimes we add some details.

Fact A.0.8. [21, Prop. 1.8.1] Let $D$ be a saturated strongly minimal structure. Let $A$ be a small subset of $D$. Let $p(x),q(y)$ be two stationary types over $A$. Let $f(x_1,x_2),g(x,y)$ be partial $A$-definable functions such that

$(i)$ for $a_1,a_2$ $A$-independent realisations of $p$, $f(a_1,a_2)$ is defined, and if $a_3 = f(a_1,a_2)$ then $a_3$ realises $p$, and $a_3$ is independent with each of $a_1,a_2$ over $A$,
(ii) for \(a, b\) independent realisations of \(p, q\) respectively, \(g(a, b)\) is defined, realises \(q\), and is independent with \(a\) over \(A\),

(iii) if \(a_1, a_2, a_3\) are \(A\)-independent realisations of \(p\), then
\[
f(f(a_1, a_2), a_3) = f(a_1, f(a_2, a_3)),
\]

(iv) if \(a_1, a_2\) realise \(p\), \(b\) realise \(q\), and \(\{a_1, a_2, b\}\) is \(A\)-independent, then
\[
g(f(a_1, a_2), b) = g(a_1, g(a_2, b)).
\]

Then there are interpretable in \(D\) a definably connected group \(H\), a (multiplicity 1) set \(X\) and a transitive action of \(H\) on \(X\), all defined over \(A\), as well as partial \(A\)-definable invertible functions \(h_1, h_2\) such that

(i) for a realising \(p\), \(h_1(a)\) realises the generic type of \(H\), and for \(b\) realising \(q\), \(h_2(b)\) realises the generic type of \(X\),

(ii) for \(a_1, a_2\) \(A\)-independent realisations of \(p\), \(h_1(f(a_1, a_2)) = h_1(a_1) \cdot h_1(a_2)\),

(iii) for independent realisations \(a\) of \(p\) and \(b\) of \(q\), \(h_2(g(a, b)) = h_1(a) \cdot h_2(b)\).

**Proposition A.0.9.** [21, Proposition 3.1] Let \(R\) be a sufficiently saturated real closed field and \(D = R(\sqrt{-1})\) its algebraic closure. Let \(G\) be a group definable in \(R\). Then there is a finite subset \(A\) of \(R\) over which \(G\) is defined, a definably connected group \(H\) definable in \(D\) over \(A\), points \(a, b, c\) of \(G\) and points \(a', b', c'\) of \(H(R)\), such that

(i) \(a \cdot b = c\) (in \(G\)) and \(a' \cdot b' = c'\) (in \(H\)),

(ii) \(acl(aA) = acl(a'A)\), \(acl(bA) = acl(b'A)\), and \(acl(cA) = acl(c'A)\),

(iii) \(a\) and \(b\) are generic points of \(G\) over \(A\) and \(a\) is independent with \(b\) over \(A\).

(iv) Similarly \(a'\) and \(b'\) are generic points of \(H\) over \(A\) and are independent with each other over \(A\).

**Proof.** Let \(n = \dim(G)\). Let \(A_0\) be a finite subset of \(R\) over which \(G\) and its group operation are defined. Since for any finite subset \(A\) of \(R\) and any \(A\)-definable subset \(X\) of \(R^n\), \(X\) has a generic point over \(A\), there are \(a, b\) \(A_0\)-independent generic points
of $G$ over $A_0$. Let $c = a \cdot b$. Then, $c$ is also a generic point of $G$ over $A_0 \cup \{a\}$ and $c \downarrow_{A_0} b$. In $R$, $c \in \text{dcl}(a, b, A_0)$ and $b \in \text{dcl}(a, c, A_0)$. Thus we initiate with three generic points of $G$ such that two of them are independent (over some set of parameters) and define the third in the substructure $R$. The key is to modify those points by points in $R$ such that two of them define the third in the structure $D$; namely, that dcl is replaced by qfdcl. This will be crucial in laying the foundations to apply Fact A.0.8.

**Lemma A.0.10.** There are a finite subset $A_2$ of $R$, containing $A_0$, with $\{a, b\}$ and $A_2$ independent over $A_0$, and tuples $a_1, b_1, c_1$ in $R$ such that $\text{acl}(a, A_2) = \text{acl}(a_1, A_2)$, $\text{acl}(b, A_2) = \text{acl}(b_1, A_2)$, $\text{acl}(c, A_2) = \text{acl}(c_1, A_2)$, $b_1 \in \text{qfdcl}(a_1, c_1, A_2)$ and $c_1 \in \text{qfdcl}(a_1, b_1, A_2)$.

**Proof.** Let $x'$ be generic of $G$ over $A_0 \cup \{a, b\}$. Let $y' = x' \cdot b$ and $z' = x' \cdot a^{-1}$. Then $z' \cdot c = y'$, and hence $\dim(y'/A_0, a, b, c, z') = 0$. By genericity of $x'$, $z'$ is generic of $G$ over $A_0 \cup \{a, b\}$. Thus we have the following.

$$\dim(a, b, c/A_0) = \dim(a, b/A_0) = 2n, \quad (A.0.1.1)$$

$$\dim(z', y', c/A_0) = \dim(c/A_0) + \dim(z'/A_0c) + \dim(y'/A_0z'c) = 2n, \quad (A.0.1.2)$$

$$\dim(a, b, z', y', c/A_0) = \dim(a, b, c/A_0) + \dim(z', y'/A_0abc) = 3n. \quad (A.0.1.3)$$

**Claim A.0.11.** Let $c'$ be any tuple in $D$ such that

$$\text{qftp}(c'/a, z', b, y', A_0) = \text{qftp}(c/a, z', b, y', A_0).$$

Then $c$ and $c'$ are interalgebraic over $A_0$.

**Proof.** Since $z' \cdot c = y'$, $a \cdot b = c$ and $\text{qftp}(c'/A_0, a, z', b, y') = \text{qftp}(c/A_0, a, z', b, y')$, then $\dim(c'/A_0, z', y') = \dim(c'/A_0, a, b) = 0$. Therefore, $\dim(c'/A_0, z', y', c)$ and $\dim(c'/A_0, a, b, c)$ are both equal to zero.
From the above, Equations (A.0.1.1) and (A.0.1.2) yield

\[
\dim (z', y', c, c'/A_0) = \dim (z', y', c/A_0) = 2n, \tag{A.0.1.4}
\]

\[
\dim (a, b, c, c'/A_0) = \dim (a, b, c/A_0) = 2n. \tag{A.0.1.5}
\]

Let \( m = \dim (c, c'/A_0) \). Then, (A.0.1.4) and (A.0.1.5) imply that

\[
\dim (z', y'/A_0, c, c') = \dim (a, b/A_0, c, c') = 2n - m.
\]

Hence,

\[
\dim (a, b, z', y', c, c'/A_0) = \dim (c', c/A_0) + \dim (a, b/A_0, c, c') + \dim (z', y'/A_0, c, c', a, b) \\
\leq m + (2n - m) + \dim (z', y'/A_0, c, c') \\
= 4n - m.
\]

By the previous equation and (A.0.1.3),

\[
3n = \dim (a, b, z', y', c/A_0) \leq \dim (a, b, z', y', c, c'/A_0) \leq 4n - m.
\]

Then, \( m \leq n \). As \( \dim (c, c'/A_0) \geq \dim (c/A_0) = n, n \leq m \leq n \). This implies that \( \dim (c'/A_0, c) = \dim (c/A_0, c') = 0 \); i.e., \( c \) and \( c' \) are interalgebraic over \( A_0 \). \( \square \)

Let \( X \) be the set of elements in \( D \) whose type in \( D \) over \( \{a, z', b, y', A_0\} \) is the same as that of \( c \) (the conjugates of \( c \) in \( D \) over \( \{a, z', b, y', A_0\} \)). Since \( c \in \text{acl} (a, z', b, y', A_0) \), \( X \) is finite. As \( D \) has elimination of imaginaries, there is a tuple \( c_1 \) in \( D \) interdefinable with \( X \) in \( D \). Since \( D \) is saturated and \( c_1 \) is fixed by every automorphism of \( D \) that fixes \( \{a, b, z', y', A_0\} \) pointwise, then \( c_1 \in \text{qfdcl} (a, z', b, y', A_0) \). As \( R \) is definably closed in \( D \), \( c_1 \) is a tuple in \( R \). Denote the concatenated sequence \( (a, z') \) by \( a_1 \), and \( (b, y') \) by \( b_1 \). Let \( A_1 = A_0 \cup \{x'\} \).
Hence, we directly have that $\text{acl}(a, A_1) = \text{acl}(a_1, A_1)$, $\text{acl}(b, A_1) = \text{acl}(b_1, A_1)$, and by Claim A.0.11, $\text{acl}(c, A_1) = \text{acl}(c_1, A_1)$. Moreover, $c_1 \in \text{qfdcl}(a, b_1, A_0)$.

Now, let $z_1$ generic of $G$ over $A_1 \cup \{a, b\}$. Let $x_1 = z_1 \cdot a$, $y_1 = z_1 \cdot c$. Hence, $x_1 \cdot b = y_1$.

Thus we have the following.

1. $x_1, b$ are generic points of $G$ over $A_0$, $x_1 \downarrow_{A_0} b$, and
2. $x_1$ is generic of $G$ over $A_0 \cup \{x', a, b\}$ since $z_1$ is. So $x_1 \downarrow_{A_0} \{x', a, b\}$. In particular, $x_1 \downarrow_{A_0, b} x'$. Also, $b \downarrow_{A_0} x'$, then $\{x_1, b\} \downarrow_{A_0} x'$.

As in the former case ($x'$ with $a, b$), consider $x'$ with $x_1, b$ and define $z'' = x' \cdot x_1^{-1}$ and $y'' = x' \cdot b$ (so $y'' = y'$), which are generic of $G$ over $A_0 \cup \{x', b\}$. And as in (A.0.1.1), (A.0.1.2), and (A.0.1.3) we have that

$$\dim(x_1, b, z'', y'', y_1/A_0) = 3n, \dim(z'', y'', y_1/A_0) = \dim(x_1, b, y_1/A_0) = 2n.$$  

Similar arguments to the proof of Claim A.0.11 show the following.

**Claim A.0.12.** Let $y'_1$ be a tuple in $D$ such that

$$\text{qftp}(y'_1/x_1, b, z'', y'', A_0) = \text{qftp}(y_1/x_1, b, z'', y'', A_0).$$

Then $y_1$ and $y'_1$ are interalgebraic over $A_0$.

And again, there is a tuple $y_2 \in R$ interdefinable in $D$ with all the conjugates in $D$ of $y_1$ over $A_0 \cup \{x_1, b, z'', y''\}$, then $y_2 \in \text{qfdcl}(x_1, b, z'', y'', A_0)$.

Denote the concatenated sequence $(x_1, z'')$ by $x_2$, and $(b, y'')$ by $b_2$. As $y'' = y'$, $b_2 = b_1$. Then, $\text{acl}(x_1, A_1) = \text{acl}(x_2, A_1)$, $\text{acl}(b, A_1) = \text{acl}(b_2, A_1)$, and by Claim A.0.12, $\text{acl}(y_1, A_1) = \text{acl}(y_2, A_1)$. Also $y_2 \in \text{qfdcl}(x_2, b_2, A_1)$.

From the definition of each of the tuples, we have that:

$$\dim(a_1, c_1, b_2, x_2, y_2/A_1) = 3n, \dim(a_1, b_2, c_1/A_1) = \dim(x_2, b_2, y_2/A_1) = 2n.$$  

And as in Claims A.0.11 and A.0.12, we have the following.
Claim A.0.13. Let $b'_2$ be a tuple in $D$ such that

$$qftp(b'_2/a_1,c_1,x_2,y_2,A_1) = qftp(b_2/a_1,c_1,x_2,y_2,A_1).$$

Then $b_2$ and $b'_2$ are interalgebraic over $A_1$.

Let $A_2 = A_1 \cup \{z_1\}$.

Again, there is a tuple $b_3 \in R$ interdefinable in $D$ with all the conjugates in $D$ of $b_2$ over $A_2 \cup \{a_1,c_1,x_2,y_2\}$, then $b_3 \in qfdcl(a_1,c_1,x_2,y_2,A_2)$. As $y_2 \in qfdcl(x_2,b_2,A_1)$ and $c_1 \in qfdcl(a_1,b_1,A_1)$, then $c_1,y_2 \in qfdcl(a_1,x_2,b_2,A_2)$.

Therefore, there are some quantifier-free $A_2 \cup \{a_1,x_2\}$-definable functions $f$ and $g$ such that $y_2 = f(b_2)$ and $c_1 = g(b_2)$.

Then for every $b'_2 \in D$ satisfying the same quantifier-free type as $b_2$ over $A_2 \cup \{a_1,x_2,y_2,c_1\}$, also $y_2 = f(b'_2)$ and $c_1 = g(b'_2)$. As $b_3$ is interdefinable in $D$ with all the conjugates of $b_2$, $y_2$ and $c_1$ are in $qfdcl(a_1,x_2,b_3,A_2)$.

Denote the concatenated sequence $(a_1,x_2)$ by $a_1$, $(y_2,c_1)$ by $c_1$, and $b_3$ by $b_1$. From the foregoing, $a_1$, $b_1$, $c_1$, and $A_2$ satisfy the conclusions of Lemma A.0.10. Furthermore, $a_1 \downarrow_{A_2} b_1$, $a_1 \downarrow_{A_2} c_1$, and $b_1 \downarrow_{A_2} c_1$. Thus we finish the proof of Lemma A.0.10. \qed

Let $A = acl(A_2) \cap R$. From now on, we work in $D$.

Below we prove that the canonical base $\sigma$ of $qftp(b_1,c_1/A_1)$ is in the quantifier-free definable closure of $A \cup \{a_1\}$, and that $\sigma$ and $a$ are interalgebraic over $A$. For this, we will see the next result.

Proposition A.0.14. Let $R$ be a real closed field, and $D$ its algebraic closure. Let $A' \subseteq R$, $A = acl(A') \cap R$ and $p = qftp(a/A)$ for some tuple $a$ in $R$. Then $p$ is stationary.

Proof. Let $q = qftp(a/acl(A'))$. Then $q$ is stationary (this is easily seen from the finite equivalence relation theorem and by the elimination of imaginaries in $D$). By [36, 2.26(iii)], $q$ does not fork over $A$ and $q|_A$ is stationary if and only if the canonical base of $q$, $cb(q)$, is contained in $qfdcl(A)$. By [36, 2.26(v)], $cb(q) \subseteq qfdcl(a,A')$. Also $cb(q) \subseteq acl^{eq}(acl(A'))$, so $cb(q) \subseteq acl(A')$ since $D$ eliminates imaginaries.
Therefore, \( cb(q) \subseteq qfdcl(a, A') \cap acl(A') \subseteq acl(A') \cap R \) since \( A' \cup \{a\} \subseteq R \) and \( R \) is definably closed in \( D \), then \( cb(q) \subseteq A \subseteq qfdcl(A) \). Hence, \( q|_A = p \) is stationary.

**Remark A.0.15.** \( qftp(a_1, b_1, c_1/A) \) and \( qftp(b_1, c_1/A, a_1) \) are both stationary.

*Proof.* Since \( A \) is relatively algebraically closed in \( R \) and \( a_1, b_1, c_1 \) are tuples in \( R \), Proposition A.0.14 yields \( qftp(a_1, b_1, c_1/A) \) is stationary. And by the same reason, \( qftp(b_1/A) \) is stationary. As \( a_1 \downarrow_A b_1 \), \( qftp(b_1/A, a_1) \) is a nonforking extension of the stationary \( qftp(b_1/A) \), then \( qftp(b_1/A, a_1) \) is also stationary. Since \( c_1 \in qfdcl(a_1, b_1, A) \), the stationarity of \( qftp(b_1/A, a_1) \) implies the stationarity of \( qftp(b_1, c_1/A, a_1) \).

Therefore, by Remark A.0.15, the canonical base \( \sigma \) of \( qftp(b_1, c_1/A, a_1) \) is defined, and thus \( \sigma \in qfdcl(a_1, A) \). And since \( R \) is definably closed in \( D \), \( \sigma \in R \).

Let \( r = qftp(\sigma/A) \), \( q_1 = qftp(b_1/A) \), and \( q_2 = qftp(c_1/A) \). We recall that we work in \( D \).

**Remark A.0.16.** \( r \) is stationary. \( \dim(r) = n. \sigma \downarrow_A b_1, \sigma \downarrow_A c_1, \) and \( \sigma \) and \( a \) are interalgebraic over \( A_0 \). \( b_1 \in qfdcl(\sigma, c_1, A) \) and \( c_1 \in qfdcl(\sigma, b_1, A) \).

*Proof.* By A.0.14, \( r \) is stationary. Now observe that as \( \sigma \in qfdcl(a_1, A) \),

\[
\dim(\sigma/A) \leq \dim(\sigma, a_1/A) = \dim(a_1/A) = n.
\]

Also, since \( a_1 \downarrow_A b_1 \),

\[
\dim(b_1/A) = \dim(b_1/A, a_1) \leq \dim(b_1/A, \sigma) \leq \dim(b_1/A),
\]

therefore \( \dim(b_1/A, \sigma) = \dim(b_1/A) \), i.e., \( \sigma \downarrow_A b_1 \).

Similarly, from \( a_1 \downarrow_A c_1 \), \( \sigma \downarrow_A c_1 \).
Since $\sigma$ is the canonical base of $\text{qftp}(b_1, c_1/A, a_1)$, $b_1 \in \text{qfdcl}(a_1, c_1, A)$ and $c_1 \in \text{qfdcl}(a_1, b_1, A)$, then $b_1 \in \text{qfdcl}(\sigma, c_1, A)$ and $c_1 \in \text{qfdcl}(\sigma, b_1, A)$. Thus,

$$\dim(c_1/A, b_1) \leq \dim(c_1, \sigma/A, b_1) = \dim(\sigma/A, b_1).$$

So,

$$n = \dim(c_1/A) = \dim(c_1/A, b_1) \leq \dim(\sigma/A, b_1) = \dim(\sigma/A).$$

Thus, $n \leq \dim(\sigma/A) \leq n$, i.e., $\dim(r) = n$. Finally, since $n = \dim(\sigma, a_1/A) = \dim(\sigma/A) + \dim(a_1/A, \sigma)$, $a_1$ and $\sigma$ are interalgebraic over $A$.

\[\Box\]

From Remark A.0.16, $c_1 \in \text{qfdcl}(\sigma, b_1, A)$ then there is some quantifier-free $A$-definable partial function $\mu$ such that $c_1 = \mu(\sigma, b_1)$. We write $\mu(\sigma, b_1)$ as $\sigma \cdot b_1$. Likewise, as $b_1 \in \text{qfdcl}(\sigma, c_1, A)$, there is some quantifier-free $A$-definable partial function $\nu$ such that $b_1 = \nu(\sigma, c_1)$ and write $\nu(\sigma, c_1)$ as $\sigma^{-1} \cdot c_1$. Note that if $\sigma' \models r$, $b'_1 \models q_1$, and $\sigma' \downarrow_A b'_1$, $\sigma' \cdot b'_1$ is well defined and $\sigma' \cdot b'_1 \models q_2$. Similarly, if $\sigma' \models r$, $c'_1 \models q_2$, and $\sigma' \downarrow_A c'_1$, $(\sigma')^{-1} \cdot c'_1$ is well defined and $(\sigma')^{-1} \cdot c'_1 \models q_1$.  

**Remark A.0.17.** Let $\sigma_1, \sigma_2 \models r$ and $b' \models q_1$ such that $b' \downarrow_A \{\sigma_1, \sigma_2\}$ and $\sigma_1 \cdot b' = \sigma_2 \cdot b'$, then $\sigma_1 = \sigma_2$.

**Proof.** Let $c'$ be the common value of $\sigma_1 \cdot b'$ and $\sigma_2 \cdot b'$. From the hypothesis and the definition of $c'$,

$$n = \dim(b', c'/A, \sigma_1, \sigma_2) = \dim(b', c'/A, \sigma_1) = \dim(b', c'/A, \sigma_2).$$

Hence, each of $\sigma_1, \sigma_2$ is the canonical base of $\dim(b', c'/A, \sigma_1, \sigma_2)$, which is a stationary type, then $\sigma_1$ and $\sigma_2$ are interdefinable in $D$ and have the same quantifier-free type over $A$, by the construction of canonical bases (see [21, 1.6]), $\sigma_1 = \sigma_2$.

\[\Box\]

Now, we will produce the germ of an invertible function from $q_1$ to $q_1$ and with this we will establish the conditions for applying the Fact A.0.8. Such germ is defined
through two $A$-independent realisations of $r$.

Let $\sigma_1, \sigma_2 \models r$ with $\sigma_1 \downarrow_A \sigma_2$ and $\sigma_1, \sigma_2 \in R$. Let $b_2 \models q_1$ such that $b_2 \downarrow_A \{\sigma_1, \sigma_2\}$ and $b_2 \in R$. Then $\sigma_1 \cdot b_2 \models q_2$ and $\sigma_1 \cdot b_2 \downarrow_A \sigma_2$. Hence, $\left(\sigma_2^{-1} \cdot (\sigma_1 \cdot b_2)\right) = b_3$ is defined and realises $q_1$. Intuitively, $(\sigma_2)^{-1} \cdot \sigma_1$ is as the germ of an invertible map from $q_1$ to $q_1$.

**Remark A.0.18.** $b_3 \in \text{qfdcl} \ (\sigma_1, \sigma_2, b_2, A)$, $b_2 \in \text{qfdcl} \ (\sigma_1, \sigma_2, b_3, A)$, each of $b_2$, $b_3$ is independent with $\{\sigma_1, \sigma_2\}$ over $A$. Also $\text{qftp} (b_2, b_3/\sigma_1, \sigma_2, A)$ is stationary.

**Proof.** The first statement follows directly from the conditions of the elements. Now, observe that $\text{qftp} (b_2/\sigma_1, \sigma_2, A)$ is stationary because $\text{qftp} (b_2/A)$ is stationary and $b_2 \downarrow_A \{\sigma_1, \sigma_2\}$. Since $b_3 \in \text{qfdcl} \ (\sigma_1, \sigma_2, b_2, A)$, the stationarity of $\text{qftp} (b_2/\sigma_1, \sigma_2, A)$ yields the stationarity of $\text{qftp} (b_2, b_3/\sigma_1, \sigma_2, A)$.

\[\Box\]

Let $\tau$ be the canonical base of $\text{qftp} (b_2, b_3/\sigma_1, \sigma_2, A)$. As $b_3 \in \text{qfdcl} \ (\sigma_1, \sigma_2, b_2, A)$ and $b_2 \in \text{qfdcl} \ (\sigma_1, \sigma_2, b_3, A)$, $b_3 \in \text{qfdcl} \ (\tau, b_2, A)$ and $b_2 \in \text{qfdcl} \ (\tau, b_3, A)$. Therefore, there are some quantifier-free $A$-definable partial functions $\mu'$ and $\nu'$ such that $b_3 = \mu'(\tau, b_2)$ and $b_2 = \nu'(\tau, b_3)$. And as before, we write $\tau \cdot b_2 = b_3$ and $\tau^{-1} \cdot b_3 = b_2$.

Let $s = \text{qftp} (\tau/A)$. Then, $s$ is stationary, and, as previously, $\tau$ can be viewed as the germ of an invertible map from $q_1$ to itself. As in A.0.17, one can prove the following.

**Remark A.0.19.** Let $\tau, \tau' \models s$ and $b' \models q_1$ such that $b' \downarrow_A \{\tau, \tau'\}$ and $\tau \cdot b' = \tau' \cdot b'$, then $\tau = \tau'$.

**Lemma A.0.20.** With the above notation, $\dim (s) = n$. $\tau$ is independent with each of $\sigma_1, \sigma_2$ over $A$.

**Proof.** In the first part of the proof we will see that $\dim (s) = n$ assuming that $\tau$ is independent with each of $\sigma_1, \sigma_2$ over $A$. After we will prove this independence part.

Since $\sigma_1, \sigma_2, \tau$ arise as canonical bases, each of $\sigma_1, \sigma_2, \tau$ is in the algebraic closure of $A$ together with the other two. This, along with $\sigma_1 \downarrow_A \tau$, yields $\dim (\sigma_1, \sigma_2, \tau/A) = \dim (\sigma_1, \sigma_2/A) = 2n$, and
\[
\dim (\sigma_1, \sigma_2, \tau/A) = \dim (\sigma_1/A) + \dim (\tau/\sigma_1, A) + \dim (\sigma_2/\sigma_1, \tau, A)
\]
\[
= n + \dim (\tau/A) + 0.
\]

Then, \(\dim (s) = n\).

Now, we will show that \(\sigma_1 \downarrow_A \tau\) and \(\sigma_2 \downarrow_A \tau\) proving that \(\tau \in \text{acl} (x, x_1, A)\) and \(\{x, x_1\}\) is independent from each of \(\sigma_1, \sigma_2\) over \(A\) for some \(x, x_1\) in \(R\).

As a matter of notation we may as well assume that \(\sigma_1 \cdot b_1 = c_1\) (i.e., \(\sigma_1 = \sigma\)).

Let \(x\) be generic of \(G\) over \(A \cup \{a, b\}\) where \(a, b, c\) are as at the beginning of the proof of A.0.9 (remember that \(a_1, b_1, c_1\) are each interalgebraic over \(A\) with \(a, b, c\) respectively). We may suppose that \(\sigma_2\) is independent from \(a, b, x\) over \(A\). Let \(y = x \cdot b\) and \(z = y \cdot c^{-1}\) where \(\cdot\) is the group operation of \(G\). So \(z = x \cdot a^{-1}\) and \(\sigma_2\) is independent from \(\{x, y, z, a, b, c\}\) over \(A\). \(\sigma_1\) is independent from \(\{z, c_1, y\}\) over \(A\) since \(\sigma_1\) is interalgebraic with \(a\) over \(A\). By the stationarity of \(r, \sigma_1\) and \(\sigma_2\) have the same quantifier-free type over \(A \cup \{z, c_1, y\}\). So, using the saturation of \(D\), we can find elements \(x_1\) and \(b_2\) in \(D\) such that \(\text{qftp} (\sigma_1, c_1, b_1, x, y, z/A)\) is equal to \(\text{qftp} (\sigma_2, c_1, b_2, x_1, y, z/A)\). It is easily seen that \(\tau \cdot b_1 = b_2\). Now, we make the following claims, which imply the desired independence statement:

(A) \(\{x_1, x\}\) is independent from each of \(\sigma_1, \sigma_2\) over \(A\).

(B) \(\tau \in \text{acl} (x, x_1, A)\).

(A) Let \(X\) be the collection of all points considered in this proof, i.e., \(X = \{\sigma_1, b, c, b_1, c_1, x, y, z, \sigma_2, x_1b_2\}\), then \(\dim (X/A) = 4n\). In addition, from the definition of \(X\) we have that \(X\) is contained in \(\text{acl} (\sigma_1, x, x_1, b_1, A)\) as well as in \(\text{acl} (\sigma_2, x, x_1, b_1, A)\). However, each of the points \(\sigma_1, \sigma_2, x, x_1, b_1\) has dimension \(n\) over \(A\), then \(\dim (\sigma_1, x, x_1/A) = \dim (\sigma_2, x, x_1/A) = 3n\) implying (A).

(B) From the former paragraph we have that

(i) \(\dim (\sigma_1, x, x_1, b_1/A) = 4n\), whereby \(b_1 \downarrow_A \{\sigma_1, x, x_1\}\).
As \( \sigma_2 \in \text{acl}(\sigma_1, x, x_1, A) \), then

(ii) \( b_1 \upharpoonright_A \{\sigma_1, \sigma_2, x, x_1\} \).

Since \( b_2 \in \text{dcl}(\tau, b_1) \), by (ii) we have that

(iii) \( \tau \) is the canonical base of \( \text{qftp}(b_1, b_2/\sigma_1, \sigma_2, x, x_1, A) \).

On the other hand, as \( y \in \text{acl}(b_1, x, A) \) and \( b_2 \in \text{acl}(y, x_1, A) \), then

(iv) \( b_2 \in \text{acl}(x, x_1, b_1, A) \).

Thus \( \text{qftp}(b_1, b_2/\sigma_1, \sigma_2, x, x_1, A) \) does not fork over \( \{x, x_1, A\} \). So by the construction of canonical bases and (iii), \( \tau \in \text{acl}(x, x_1, A) \), which concludes the proof of Lemma A.0.10.

\[ \square \]

Recalling the notation from A.0.18, we have that \( \tau \in \text{qfdcl}(\sigma_1, \sigma_2, A) \), then there is some quantifier-free \( A \)-definable partial function \( \xi \) such that \( \tau = \xi(\sigma_1, \sigma_2) \) that we write \( \tau = \sigma_2^{-1} \cdot \sigma_1 \). By A.0.18, \( \tau \upharpoonright_A b_2 \) and \( \tau \upharpoonright_A b_3 \). Recall that since \( b_3 \in \text{qfdcl}(\tau, b_2, A) \) and \( b_2 \in \text{qfdcl}(\tau, b_3, A) \), we write \( \tau \cdot b_2 = b_3 \) and \( \tau^{-1} \cdot b_3 = b_2 \).

**Lemma A.0.21.** There is a function \( f \), (quantifier-free) definable in \( D \) with parameters in \( A \), such that for \( A \)-independent realisations \( \tau_1 \) and \( \tau_2 \) of \( s \), \( f(\tau_1, \tau_2) \) realises \( s \) and is independent over \( A \) with each of \( \tau_1, \tau_2 \), and moreover for \( b' \) realising \( q_1 \) independent of \( \{\tau_1, \tau_2\} \) over \( A \), \( f(\tau_1, \tau_2) \cdot b' = \tau_1 \cdot (\tau_2 \cdot b') \).

**Proof.** Let \( \sigma_2 \) realise \( r \) such that \( \sigma_2 \upharpoonright_A \{\tau_1, \tau_2, b'\} \). By Lemma A.0.20, there are realisations \( \sigma_1, \sigma_3 \) of \( r \) such that \( \tau_1 = \sigma_1^{-1} \cdot \sigma_2 \) and \( \tau_2 = \sigma_2^{-1} \cdot \sigma_3 \). Then \( \tau_2 \cdot b' \models q_1 \) and \( \tau_2 \cdot b' \upharpoonright_A \tau_1 \), thus \( \tau_1 \cdot (\tau_2 \cdot b') \) is well-defined, and equals say \( c' \). On the other hand, from A.0.20 \( \sigma_1 \upharpoonright_A \sigma_3 \) and \( b' \upharpoonright_A \{\sigma_1, \sigma_3\} \). Then \( \sigma_1^{-1} \cdot \sigma_3 = \tau_3 \) realises \( s \) and clearly \( \tau_3 \) is the canonical base of \( \text{qftp}(b', c'/\tau_1, \tau_2, A) \), and also \( \tau_3 \cdot b' = c' \). Therefore, for some (quantifier-free) \( A \)-definable function \( f \), \( f(\tau_1, \tau_2) = \tau_3 \), which finishes the proof.

\[ \square \]
Now we have the conditions for applying Fact A.0.8, where the types \( q \) and \( p \) in Fact A.0.8 are the stationary types \( q_1 = \text{qftp}(b_1/A) \) and \( s = \text{qftp}(\tau/A) \), respectively. The function \( g(\tau, b_2) = \tau \cdot b_2 = b_3 \) that results from the fact that \( b_3 \in \text{qfdcl}(\tau, b_2, A) \) is the action of the set of realisations of \( s \) on the set of realisations of \( q_1 \), which acts as the function \( g \) of Fact A.0.8. And the function \( f \) in Fact A.0.8 is provided by Lemma A.0.21.

Note that of all the above, the only hypothesis of Fact A.0.8 that remains to prove is the associativity of \( f \). Then, let \( \{\tau_1, \tau_2, \tau_3\} \) \( A \)-independent realisations of \( s \). Let \( b \) realise \( q_1 \) independently from \( \{\tau_1, \tau_2, \tau_3\} \) over \( A \). Then by Lemma A.0.21, \( \tau_1 \cdot (\tau_2 \cdot (\tau_3 \cdot b)) = f(\tau_1, f(\tau_2, \tau_3)) \cdot b = f(f(\tau_1, \tau_2), \tau_3) \cdot b \). Then by A.0.19, \( f(\tau_1, f(\tau_2, \tau_3)) = f(f(\tau_1, \tau_2), \tau_3) \).

Now, let \( H, X \) and \( h_1, h_2 \) be as given by A.0.8. We can assume that \( h_1, h_2 \) are both the identity function. Thus already \( s \) is the generic type of \( H, q_1 \) is the generic type of \( X \), and for \( \tau_1, \tau_2 \) \( A \)-independent realisations of \( s \), the product \( \tau_1 \cdot \tau_2 \) in the group \( H \) is exactly \( f(\tau_1, \tau_2) \). Also the notation \( \Lambda : H \times X \to X : (\tau, b) \mapsto \Lambda(\tau, b) \) for the group action of \( H \) on \( X \) agrees with the earlier notation \( \mu(\tau, b) \) when \( \tau \) and \( b \) are independent realisations of \( s, q_1 \), respectively.

We now complete the proof of Proposition A.0.9. Let \( \sigma, b_1, c_1 \) be as fixed prior to Remark A.0.16. Let \( \sigma_1 \in R \) such that \( \text{qftp}(\sigma_1/A) = r \) and \( \sigma_1 \downarrow_A \{\sigma, b_1, c_1\} \).

Then \( \nu(\sigma_1, c_1) = \sigma_1^{-1} \cdot c_1 \in R \), realises \( q_1 \) and is independent from \( \sigma_1 \) over \( A \). Similarly \( \xi(\sigma_1, \sigma) = \sigma_1^{-1} \cdot \sigma \in H(R) \), realises \( s \) and is independent from \( \sigma_1 \) over \( A \). Denote \( \sigma_1^{-1} \cdot c_1 \) by \( c_2 \), and \( \sigma_1^{-1} \cdot \sigma \) by \( \tau \). Let \( A_1 = \text{acl}(\sigma_1, A) \cap R \). Then we have that \( \dim(a, b, c/A_1) = 2n, \text{acl}(a, A_1) = \text{acl}(\tau, A_1) \), and \( \text{acl}(c, A_1) = \text{acl}(c_2, A_1) \).

We make a second extension of a similar nature. Let \( \tau_1 \in R \) realise \( s \) independently from \( \{\tau, b_1, c_2\} \) over \( A_1 \). Let \( \tau_2 = \tau \cdot \tau_1, b_2 = \Lambda(\tau_1^{-1}, b_1) \). So \( \Lambda(\tau_2, b_2) = c_2 \), and also, \( b_2 \downarrow_{A_1} \{\tau, b_1, c_2\} \). Let \( A_2 = \text{acl}(b_2, A_1) \cap R \). Hence, we have that \( \dim(a, b, c/A_2) = 2n, \text{acl}(a, A_2) = \text{acl}(\tau, A_1), \text{acl}(b, A_2) = \text{acl}(\tau_1, A_2), \text{acl}(c, A_2) = \text{acl}(\tau_2, A_2), \) and \( \tau \cdot \tau_1 = \tau_2 \).

Let \( a' = \tau, b' = \tau_1 \), and \( c' = \tau \cdot \tau_1 \).

Finally, we may shrink \( A_2 \) to a finite set over which \( G \) and \( H \) are defined and with the properties just obtained. This completes the proof of Proposition A.0.9. \( \square \)
Bibliography


105


106


